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PROPER RISK AVERSION

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We introduce and investigate a significant behavioral condition on utility functions for wealth, which we call proper risk aversion, namely that an undesirable lottery can never be made desirable by the presence of an independent, undesirable lottery. (Independent random background wealth is allowed.) One consequence is that offering insurance and other forms of hedging to proper investors can only encourage them to accept other, independent risks. Properness implies decreasing risk aversion and, much less immediately, several stronger intuitive conditions on certainty equivalents and risk premiums. We prove properness for all mixtures of risk-averse exponential functions and hence (by complete monotonicity) of all risk-averse power and logarithmic functions. We derive analytical necessary and sufficient conditions, which seem to be unavoidably complicated, and local necessary conditions, alas not sufficient, which are tractable.

KEYWORDS: Risk aversion, lotteries, utility, insurance, proper risk aversion, complete monotonicity, independent risks.

1. INTRODUCTION

AN INDIVIDUAL FINDS each of two independent monetary lotteries undesirable. If he is required to take one, should he not continue to find the other undesirable? If in all such cases he would, he displays a property we call *proper risk aversion*, or *properness.* Here we investigate this property for both fixed and variable background wealth. Conveniently, properness guarantees decreasing (possibly constant) risk aversion as well. Given decreasing risk aversion, it is equivalent to the condition that two or more independent, individually undesirable lotteries cannot be desirable together.

Many decision makers think at first that they do not wish to be proper. Thus Samuelson (1963) observed that individuals would reject a bet offering +\$200 or −\$400 on the toss of a coin, but would accept a large number of such bets. He suggested that such individuals might be misinterpreting the Law of Large Numbers, underestimating the length and importance of the lower tail of the distribution of the sum of many fair bets.[2]

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[2] Alternative explanations for this phenomenon relate to the well-chronicled inconsistencies individuals display when making even modestly complex decisions under uncertainty. For example, Tversky and Bar-Hillel (1983) explain the phenomenon of accepting multiple independent gambles that are unacceptable singly on the basis of prospect theory and framing theory. In an experiment, Stanford undergraduates rejected a gamble in favor of a certainty, but preferred the gamble when a fixed amount was subtracted from (or added to) each outcome. They concluded: "These observations reject a utility function that is based on final asset position in favor of a theory where the carriers of value are gains and losses defined relative to some reference point" (1983, p. 717).

Samuelson's major result was that, if a bet would be rejected at every wealth level, so would any predetermined number of such bets, under expected utility.[3] But rejection at every wealth is a severe restriction. Properness, by contrast, only requires that the bets be rejected given current wealth (or its distribution), and permits them to be different as well.[4]

In the same spirit that risk aversion is a desirable property for a univariate utility function, so too is proper risk aversion, we believe. Once the property is understood, many real-world decision makers may wish to require it of the utility functions they employ to guide their decisions.

Proper risk aversion should have important consequences in application. For instance, insuring against one risk (i.e. rejecting one lottery) will make a proper individual readier to accept other independent risks. Therefore, if investors are proper, the opening of new options and futures markets could stimulate activity on—rather than drain capital from—competitive markets, such as those for debt instruments, by offering new opportunities to opt out of undesirable lotteries.

The next two sections define proper risk aversion more fully and show that a number of apparently stronger conditions on certainty equivalents and risk premiums follow and are therefore equivalent. Section 4 proves that any mixture of risk-averse exponential utilities is proper. Analytical necessary and sufficient conditions are derived in Section 5, but are somewhat complicated, involving three or more levels of wealth, while the truly local conditions are simple but not globally sufficient, as we shall find in Section 6. Fortunately, the mixtures of exponentials (which are closely related to Laplace transforms and complete monotonicity) provide a rich and tractable family of proper utilities. It includes, among others, all risk-averse utilities of exponential, power, and logarithmic form or, equivalently, hyperbolic absolute risk aversion, and hence all mixtures thereof.

2. DEFINITIONS

Denote a decision maker's initial wealth by w if it is certain, by $\tilde{w}$ if it is uncertain. Let $\tilde{x}$ and $\tilde{y}$ be possible additional risks. Assume that the decision maker has a probability distribution under which $\tilde{w}$, $\tilde{x}$, and $\tilde{y}$ are independent. Assume that he has a von Neumann-Morgenstern utility function u, that is, $\tilde{w}_1$ is not preferred to $(\preccurlyeq)\,\tilde{w}_2$ iff $Eu(\tilde{w}_1) \leqslant Eu(\tilde{w}_2)$. Recall that "risk averse" is defined by the condition $\tilde{w} \preccurlyeq E\tilde{w}$ for all $\tilde{w}$ and is equivalent to concavity of u. Also an intuitively natural definition of "decreasingly risk averse" (others are trivially equivalent) is that a certain decrease in certain wealth never makes an undesirable gamble desirable, i.e.,

$$(1) \qquad w+\tilde{x}+y \preccurlyeq w+y \quad \text{whenever} \quad w+\tilde{x} \preccurlyeq w \quad \text{and} \quad y<0.$$

[3] This is clear by an intuitive induction argument: you would reject the last bet no matter how the others turned out; the second to last becomes last, etc. Indeed you would reject even if allowed to choose sequentially among any number of such bets and to stop at will, subject only to a lower limit on your loss position at all times, or to any of the other conditions semimartingale theory has provided to prevent St. Petersburg-type paradoxes (Doob, 1953).

[4] Thus properness permits the possibility that a currently unacceptable bet becomes desirable if a second independent play is allowed contingent on a favorable outcome of the first.

Several analytical conditions on u have been shown equivalent and useful (Pratt, 1964, 1982a; Arrow, 1971), including decreasing "local risk aversion" $r=-u''/u'$.

Replacing y in (1) by an undesirable gamble $\tilde{y}$ with w still certain leads to the definition: u is *fixed-wealth proper* if

$$(2) \qquad w+\tilde{x}+\tilde{y} \preccurlyeq w+\tilde{y} \quad \text{whenever} \quad w+\tilde{x} \preccurlyeq w \quad \text{and} \quad w+\tilde{y} \preccurlyeq w.$$

In other words, if lotteries $\tilde{x}$ and $\tilde{y}$ are individually unattractive, the compound lottery offering both together is less attractive than either alone.

We call *u proper* if the same condition holds for uncertain $\tilde{w}$ also, that is, if

$$(3) \qquad \tilde{w}+\tilde{x}+\tilde{y} \preccurlyeq \tilde{w}+\tilde{y} \quad \text{whenever} \quad \tilde{w}+\tilde{x} \preccurlyeq \tilde{w} \quad \text{and} \quad \tilde{w}+\tilde{y} \preccurlyeq \tilde{w}.$$

Obviously proper implies fixed-wealth proper, and fixed-wealth proper implies decreasing risk aversion. Decreasing risk aversion without even fixed-wealth properness is exemplified by any utility with positive decreasing r not satisfying the local necessary condition (28) of Section 6.[5] The necessary and sufficient conditions of Section 5 suggest that fixed-wealth proper does not imply proper, but we know of no example.

If we assume decreasing risk aversion, then we shall see that (3) is equivalent to

$$(4) \qquad \tilde{w}+\tilde{x}+\tilde{y} \preccurlyeq \tilde{w} \quad \text{whenever} \quad \tilde{w}+\tilde{x} \sim \tilde{w} \sim \tilde{w}+\tilde{y},$$

and (2) to the same condition for certainties $\tilde{w}=w$.[6] Although this condition suffices to verify properness (intuitively or mathematically), consequences may be easier to derive from (2) or (3) or still stronger implications stated below.

If (3) holds for a particular $\tilde{w}$ we call *u proper at $\tilde{w}$*. Obviously proper equals proper at all $\tilde{w}$ and fixed-wealth proper equals proper at all w.

3. IMPLICATIONS FOR CERTAINTY EQUIVALENTS AND POSITIVELY ASSOCIATED RISKS

In this section we show that properness is equivalent to (4) plus decreasing risk aversion, and to some natural conditions on certainty equivalents. (These results are unexpectedly delicate, although their counterparts for decreasing risk aversion are tautologous or obvious.) We also relax the assumption of independence.

[5] In particular, the decreasingly risk-averse utility having specified values at four or more points and maximum or minimum value elsewhere has stepwise decreasing r (Meyer and Pratt, 1968) and hence is proper only in the special case of constant risk aversion. Thus, although the range of possible values of u allowed by decreasing risk aversion is typically already very narrow (as shown by numerical evidence obtained via Schlaifer, 1971), properness narrows it still further.

[6] This weakens the conclusion of (3) (preceding "whenever") to undesirability and strengthens the premise (following "whenever") to indifference. If decreasing risk aversion is assumed, either change makes the other trivial. If not, either change gives a version of (3) which is strictly weaker and does not even imply decreasing risk aversion, as may be shown by the example of quadratic utility. With decreasing risk aversion these weaker versions of (3) would make natural definitions of properness.

Define the *certainty equivalent* $C(\tilde{x}, \tilde{z})$ of a gamble $\tilde{x}$ in the presence of another gamble $\tilde{z}$ (including initial wealth) as the sure amount to which $\tilde{x}$ is indifferent: $\tilde{z}+\tilde{x} \sim \tilde{z}+C(\tilde{x}, \tilde{z})$. Letting $C(\tilde{x})$ denote $C(\tilde{x}, \tilde{w})$ for convenience, we can write condition (3) as

(5) $$C(\tilde{x}+\tilde{y}) \leqslant C(\tilde{y}) \quad \text{whenever} \quad C(\tilde{x}) \leqslant 0 \quad \text{and} \quad C(\tilde{y}) \leqslant 0,$$

and the conclusion is immediately equivalent to $C(\tilde{x}, \tilde{y}+\tilde{w}) \leqslant 0$. Replacing the last two inequalities in (5) by equalities gives (4).

The following conditions have stronger conclusions but turn out to be equivalent:

(6) $$C(\tilde{x}+\tilde{y}) \leqslant C(\tilde{x})+C(\tilde{y}) \quad \text{whenever} \quad C(\tilde{x}) \leqslant 0 \quad \text{and} \quad C(\tilde{y}) \leqslant 0;$$

(7) $$C(\tilde{x}, \tilde{y}+\tilde{w}) \leqslant C(\tilde{x}, \tilde{w}) \quad \text{whenever} \quad C(\tilde{x}, \tilde{w}) \leqslant 0 \quad \text{and} \quad C(\tilde{y}, \tilde{w}) \leqslant 0.$$

We shall see below that proper preference implies

(8) $$\tilde{w}+\tilde{x}+\tilde{y} \preccurlyeq \tilde{w}+C(\tilde{x})+\tilde{y} \preccurlyeq \tilde{w}+C(\tilde{x})+C(\tilde{y})$$

whenever $C(\tilde{x}) \leqslant 0$ and $C(\tilde{y}) \leqslant 0$. Since the first preference here is (7) and the preference between the outer terms is (6), this says that the conclusion is stronger in (7) than in (6).[7]

The conclusions can also be strengthened to allow any kind of positive association or correlation between $\tilde{x}$ and $\tilde{y}$ that renders $\tilde{x}+\tilde{y}$ less desirable than if they were independent. Thus we can allow Epstein and Tanney's (1980) positive generalized correlation, which they show equivalent to the simple condition $P(\tilde{x} \leqslant x, \tilde{y} \leqslant y) \leqslant P(\tilde{x} \leqslant x)P(\tilde{y} \leqslant y)$ for all x, y, that is, no events $\tilde{x} \leqslant x$ and $\tilde{y} \leqslant y$ have positive correlation.[8] Less restrictive but less simple allowable conditions are that $\tilde{x}+\tilde{y}$ is second- or third-order dominated by $\tilde{x}'+\tilde{y}'$ where $\tilde{x}'$ and $\tilde{y}'$ are independent with the same marginal distributions as $\tilde{x}$ and $\tilde{y}$. These are equivalent to preference for all concave or decreasingly risk averse utilities, since the means are equal (Vickson, 1975). Indeed, preference for all fixed-wealth proper utilities suffices, since the conclusions follow for all proper utilities. These dominance and preference conditions can be regarded as types of positive association.[9]

Dependence between $\tilde{w}$ and $(\tilde{x}, \tilde{y})$ seems difficult to allow, unfortunately, because even when preference is proper, risk aversion can easily be decreased by either a stochastic decrease or added noise (mean-preserving spread) in $\tilde{w}$.[10]

[7] It is easy to write all the conditions in terms of risk premiums also. In particular, (6) and (7) say that, whenever $\tilde{x}$ and $\tilde{y}$ are independent with risk premiums no less than their expected values, the risk premium of $\tilde{x}+\tilde{y}$ is no less than the sum of their risk premiums and the risk premium of $\tilde{x}$ is no less with $\tilde{y}$ present than without.

[8] Using correlation-increasing transformations analogous to mean-preserving spreads, Epstein and Tanney (1980) give a general and natural definition of greater generalized correlation, and show its equivalence to $P(\tilde{x} \leqslant x, \tilde{y} \leqslant y) \leqslant P(\tilde{x}' \leqslant x, \tilde{y}' \leqslant y)$ and to preference for all bivariate utilities of a type that includes all concave univariate utilities as a special case.

[9] Positive and greater second-order, third-order, proper, and fixed-wealth proper correlation could be defined accordingly, but analytical conditions equivalent to the latter two are probably complex, and none are invariant to rescaling of $\tilde{x}$ and $\tilde{y}$ individually.

[10] We consider the implied behavior normal and reasonable (Pratt, 1982b), but others disagree (Ross, 1981; Machina, 1982; Epstein, 1985).

To exemplify both, suppose $\tilde{w}$ has two possible values w_1 and w_2, with $w_1 < w_2$, and suppose $\tilde{x} = 0$ when $\tilde{w} = w_1$, while respectively either $\tilde{x} = -\delta < 0$ or $\tilde{x} = \pm\delta$ with equal probability when $\tilde{w} = w_2$. Then for certain values of w_1, w_2, and δ, the "derived" utility function $U(y) = Eu(\tilde{w} + y)$, which applies to risks independent of $\tilde{w}$, may easily be more risk averse than $V(y) = Eu(\tilde{w} + \tilde{x} + y)$, because changes around w_2 have less relative importance to the former than changes around $w_2 - \delta$ to the latter.

Adding some details, we summarize the main equivalences:

THEOREM 1: *For preferences in accord with expected utility, the conditions* (4) *plus decreasing risk aversion,* (6), *and* (7) *are each equivalent to proper risk aversion,* (3), *if they hold for all independent $\tilde{w}$, $\tilde{x}$, and $\tilde{y}$, to proper risk aversion on an interval if $\tilde{w}$, $\tilde{w} + \tilde{x}$, $\tilde{w} + \tilde{y}$, and $\tilde{w} + \tilde{x} + \tilde{y}$ are restricted to this interval. The same equivalences hold for fixed-wealth properness if $\tilde{w}$ is restricted to certainties w. The independence condition can be replaced by the condition that $\tilde{x}$ and $\tilde{y}$ are jointly independent of $\tilde{w}$ and positively associated in any of the senses defined above.*

A significant fact[11] used in the proof is that decreasing risk aversion, property (1), implies the same property for uncertain $\tilde{w}$,

(9) $$\tilde{w} + \tilde{x} + y \preccurlyeq \tilde{w} + y \quad \text{whenever} \quad \tilde{w} + \tilde{x} \preccurlyeq \tilde{w} \quad \text{and} \quad y < 0.$$

Three remarks: First, the foregoing result implies that (9), the generalization of (1) to uncertain $\tilde{w}$, is equivalent to (1), unlike (2) and (3), the generalizations of (1) and (9) to uncertain $\tilde{y}$. Second, this equivalence says that u is decreasingly risk averse iff every utility function derived from it is. Third, u is proper iff every utility function derived from u is fixed-wealth proper.

PROOF OF THEOREM 1: By the third remark, we may assume independence. It is immediate that (6) or (7) $\Rightarrow$ (3) $\Rightarrow$ (4) plus decreasing risk aversion. To show that (4) plus decreasing risk aversion $\Rightarrow$ (6), let $x = C(\tilde{x})$ and $y = C(\tilde{y})$, and suppose $x \leqslant 0$ and $y \leqslant 0$. Since $\tilde{w} + \tilde{x} \sim \tilde{w} + x$, replacing $\tilde{w}$ by $\tilde{w} + x$ and $\tilde{x}$ by $\tilde{x} - x$ in (9) and then exchanging x and y gives

(10) $$\tilde{w} + x + y + \tilde{x} - x \preccurlyeq \tilde{w} + x + y;$$

(11) $$\tilde{w} + y + x + \tilde{y} - y \preccurlyeq \tilde{w} + y + x.$$

Applying (4) at $\tilde{w} + x + y$ to these relations gives $\tilde{w} + \tilde{x} + \tilde{y} \preccurlyeq \tilde{w} + x + y$, which is equivalent to (6). To complete the proof, we show that (3) $\Rightarrow$ (7). Since $\tilde{w} + x + \tilde{y} \preccurlyeq \tilde{w} + x$ by (9) or (11), applying (3) to $\tilde{w} + x$, $\tilde{x} - x$, and $\tilde{y}$ gives $\tilde{w} + \tilde{x} + \tilde{y} \preccurlyeq \tilde{w} + x + \tilde{y}$, which is equivalent to (7). We have also proved (8). *Q.E.D.*

[11] For discrete $\tilde{w}$ this implication is implicit but unrecognized in Pratt (1964, Theorem 5). It and its generalization to "more risk averse" are the results of Kihlstrom, Romer, and Williams (1981), also given independently by Nachman (1980). Later Pratt (1982a) offered several shorter proofs of this and other generalizations.

Recall that, for exponential utility, preferences among risks are unaffected by wealth (since $-e^{-c(w+a)}$ is a positive multiple of $-e^{-cw}$). It is known that mixtures of concave exponential utilities have decreasing risk aversion (Pratt, 1964, 1982a). We show here that they are also proper. The most general mixture of exponential utilities is

$$u(w)=\int_0^\infty [g(s)-e^{-sw}]\,dF(s) \tag{12}$$

where g is an arbitrary function, F is nondecreasing, and e^{-sw} is to be replaced by w when $s=0$. Differencing eliminates most of the arbitrariness in g and gives the equivalent form

$$u(w)=u(w_1)+\int_0^\infty [e^{-sw_1}-e^{-sw}]\,dF(s) \tag{13}$$

for any w_1 where $u(w_1)$ is finite. Assuming convergence at more than one point, one can show by monotonicity and concavity in w (cf., e.g., Feller, 1966, p. 409) that (13), and hence (12), gives a finite u with positive odd derivatives and negative even derivatives on some interval (w_0,∞), possibly $(-\infty,\infty)$, and $u(w)=-\infty$ for $w<w_0$. A positive function with positive even derivatives and negative odd derivatives is called "completely monotone."[12] We shall call any utility of the form (12) completely proper and state the following theorem.

THEOREM 2: *A completely proper utility function is proper everywhere that it is finite.*

A proof directly from (12) is given below. To our surprise, we found no help in either classical results on complete monotonicity or the necessary and sufficient conditions for properness given in the next section.

Complete properness is obviously preserved under mixing and under changes of origin and scale of wealth, by (12), and the family of completely proper utilities is very rich. One-point distributions F obviously give exponential functions. The risk-averse power functions $u(w)\sim d(w-a)^d$, $d\leq 1$, and $\log(w-a)$, the scaled limit at $d=0$, are completely proper on (a,∞); they satisfy (13) with $dF(s)\sim s^{-d-1}e^{as}\,ds$. Thus all mixtures of these exponential, power, and logarithmic forms are completely proper.[13]

[12] According to a classical theorem of S. Bernstein, a function is completely monotone on (w_0,∞) iff it is the Laplace transform of a measure F on $[0,\infty)$ and the Laplace transform $\int_0^\infty e^{-sw}\,dF(s)$ is finite for $w>w_0$ (Feller, 1966, pp. 415–416). (No such representation is available for complete monotonicity on a finite interval.) We are grateful to Richard F. Meyer for emphatically conjecturing the importance of complete monotonicity in this context and mentioning Bernstein's Theorem, thereby rekindling our interest in mixtures of exponential utilities. Specifically u has the form (12) or (13) on (w_0,∞) iff u' is completely monotone on (w_0,∞). This and some other aspects of exponential mixtures are discussed by Brockett and Golden (1985).

[13] Since the power functions have derivatives of alternating sign they must be exponential mixtures.

Two-point distributions F give $u(w)=a-b\,e^{-sw}-c\,e^{-tw}$, where b, c, s, and t are nonnegative constants and w replaces $-e^{-sw}$ if $s=0$.[14] The case $c,\ t<0$ produces an illustration that the form (12) is not necessary and derivatives with alternating signs not sufficient for properness on a finite interval.

EXAMPLE 2: For $s>t>0$ and $b, c>0$, the utility $a-b\,e^{-sw}+c\,e^{tw}$ is proper and risk averse on the interval

$$(14)\qquad w_0=\frac{1}{s+t}\log\left(\frac{b}{c}\right)<w<\frac{1}{s+t}\log\left(\frac{b}{c}\frac{s^2}{t^2}\right)=w_1,$$

but not of the form (12), and it is improper below w_0 although its derivatives alternate in sign for $w<w_1$. It is increasing and its risk aversion is decreasing everywhere. It is risk seeking above w_1.

PROOF OF EXAMPLE 2: The signs of the derivatives and the formula $r(w)=-t+bs(s+t)/(bs+ct\,e^{sw+tw})$ for the local risk aversion are easy to obtain. Obviously r is decreasing. It is straightforward but tedious to show that the necessary local condition for properness, (28) below, fails for $w<w_0$. To show properness on (w_0, w_1), let $k=(c/b)E\,e^{s\tilde{w}+t\tilde{w}}$, let $b=E\,e^{s\tilde{w}}$ without loss of generality, and observe that

$$(15)\qquad Eu(\tilde{w}+\tilde{x})-Eu(\tilde{w})=-Es^{-s\tilde{x}}+kE\,e^{t\tilde{x}}+1-k.$$

If $\tilde{w}+\tilde{x}\sim\tilde{w}\sim\tilde{w}+\tilde{y}$, then $E\,e^{-s\tilde{x}}=kE\,e^{t\tilde{x}}+1-k$ and similarly for $\tilde{y}$, and substitution gives

$$(16)\qquad Eu(\tilde{w}+\tilde{x}+\tilde{y})-Eu(\tilde{w})=k(1-k)(E\,e^{t\tilde{x}}-1)(E\,e^{t\tilde{y}}-1).$$

For $\tilde{w}+\tilde{x}\leqslant w_1$, risk aversion implies $E\tilde{x}\geqslant 0$ and hence $E\,e^{t\tilde{x}}\geqslant e^{tE\tilde{x}}\geqslant 1$. Similarly $E\,e^{t\tilde{y}}\geqslant 1$ for $\tilde{w}+\tilde{y}\leqslant w_1$. Since $k>1$ for $\tilde{w}>w_0$, (16) is $\leqslant 0$ and the condition (4) for properness is satisfied on the interval (w_0, w_1). *Q.E.D.*

PROOF OF THEOREM 2: If u has the form (12), then so does every derived utility $U(x)=Eu(\tilde{w}+x)$. Hence it suffices to show that every u of the form (12) is proper at 0. Suppose $\tilde{x}\leqslant 0$ and $\tilde{y}\leqslant 0$. Let $x(s)$ and $y(s)$ be the certainty equivalents of $\tilde{x}$ and $\tilde{y}$ for constant risk aversion s, that is,

$$(17)\qquad e^{-sx(s)}=E\,e^{-s\tilde{x}}\quad\text{and}\quad e^{-sy(s)}=E\,e^{-s\tilde{y}};$$

$x(s)$ and $y(s)$ are decreasing in s (Pratt, 1964, Theorem 1). Except in trivial cases, there exist constants c and d such that $x(c)=y(d)=0$. Exchange $\tilde{x}$ and $\tilde{y}$ if necessary to make $c\geqslant d$. Then one can show that, because $x(s)$ and $y(s)$ are decreasing,

$$(18)\qquad e^{-sx(s)}\lesseqqgtr 1\quad\text{and}\quad e^{-sy(s)}\lesseqqgtr e^{-cy(c)}\quad\text{for}\quad s\lesseqqgtr c.$$

[14] This family is convenient because assessments of three certainty equivalents just determine the five constants (two being arbitrary scaling, of course). Three suitably linked 50-50 lotteries give five equally spaced values of u. Schlaifer (1971) provides a computer program for this case. He allows $c,\ t<0$ if required. This led us to Example 2.

By (13) with $w_1=0$ and (7),

$$(19)\qquad Eu(\tilde{x})-u(0)=\int_0^\infty E[1-e^{-s\tilde{x}}]\,dF(s)=\int_0^\infty [1-e^{-sx(s)}]\,dF(s)\leqslant 0$$

and similarly for $\tilde{y}$. We then have

$$(20)\qquad Eu(\tilde{x}+\tilde{y})-u(0)=\int_0^\infty E[1-e^{-s\tilde{x}-s\tilde{y}}]\,dF(s)$$

$$=\int_0^\infty [1-e^{-sx(s)-sy(s)}]\,dF(s)$$

by independence and (17),

$$\leqslant\int_0^\infty [e^{-sy(s)}-e^{-sx(s)-sy(s)}]\,dF(s)\quad \text{by (19) for } \tilde{y},$$

$$\leqslant e^{-cy(c)}\int_0^\infty [1-e^{-sx(s)}]\,dF(s)\leqslant 0\quad \text{by (18) and (19).}$$

Q.E.D.

5. ANALYTICAL NECESSARY AND SUFFICIENT CONDITIONS

In this section we derive analytical necessary and sufficient conditions for a utility function u to be proper or fixed-wealth proper. While they are harder to verify than one would hope, they appear impossible to simplify, suggesting that any simple sufficient conditions must be stronger than necessary.

We first reduce the proper problem to the fixed-wealth proper one by refining our third remark to Theorem 1.

THEOREM 3: *The utility function u is proper iff the derived utility function $U(x)=Eu(\tilde{w}+x)$ is proper at 0 for every three-point distribution of $\tilde{w}$.*

PROOF: Condition (3) is equivalent to max $EU(\tilde{x}+\tilde{y})-U(\tilde{y})\leqslant 0$ under the constraints $EU(\tilde{x})\leqslant U(0)$ and $EU(\tilde{y})\leqslant U(0)$. If the condition fails, it fails at a finite distribution of $\tilde{w}$. The expectations are linear in the probabilities of the possible values of $\tilde{w}$. Hence so are the maximand and the two constraints. The only other constraint is that the probabilities must sum to 1, which is also linear. Hence the maximum is achieved at a finite distribution with at most three nonzero probabilities. The conclusion follows. *Q.E.D.*

COROLLARY: *The utility u is proper iff $U(w)=u(w)+au(w+x)+bu(w+y)$ is a fixed-wealth proper function of w for all $a\geqslant 0$, $b\geqslant 0$, x, and y.*

To derive a necessary and sufficient condition for u to be fixed-wealth proper, we shall show that the "worst case" is that $\tilde{x}$ and $\tilde{y}$ have two-point distributions, each having probability close to 1 on a value close to w, and that the following condition results.

THEOREM 4: *A decreasingly risk-averse utility function u is fixed-wealth proper iff, for all w, x, and y, with* $\Delta(x)=u(x)-u(w)$,

$$\Delta(x)u'(y)+\Delta(y)u'(x)\geqslant\Delta(x+y-w)u'(w)+\Delta(x)\Delta(y)u''(w)/u'(w). \tag{21}$$

The proof reveals that without decreasing risk aversion, (21) still follows from even the weakest condition for properness at w, (4), but it is no longer sufficient.[15] The proof of sufficiency fails because (4) is too weak and (3) is not binding.

PROOF: Without loss of generality, let $w=0$ and $u(0)=0$. Then condition (4) becomes $Eu(\tilde{x}+\tilde{y})\leqslant 0$ if $Eu(\tilde{x})=0=Eu(\tilde{y})$. Since the distributions of $\tilde{x}$ and $\tilde{y}$ each appear in just one constraint, an argument like that in the preceding proof applied to each in turn shows that (4) fails, if at all, at two-point distributions, say $P(\tilde{x}=x_i)=p_i$, $P(\tilde{y}=y_i)=q_i$ for $i,j=0,1$, $p_0+p_1=q_0+q_1=1$, $x_0<0<x_1$, $y_0<0<y_1$. The condition in this case is

$$\sum_{i,j} p_iq_ju(x_i+y_j)\leqslant 0 \quad \text{if} \quad \sum_i p_iu(x_i)=0=\sum_j q_ju(y_j). \tag{22}$$

Dividing by $p_0u(x_0)=-p_1u(x_1)$ and by $q_0u(y_0)=-q_1u(y_1)$ reduces this to

$$\sum_{ij} \operatorname{sgn}(x_i)\operatorname{sgn}(y_j)u(x_i+y_j)/u(x_i)u(y_j)\leqslant 0. \tag{23}$$

For any function f which is continuous at 0, letting $x_i\to 0$ shows that

$$f(x_1)\leqslant f(x_0) \quad \text{for} \quad x_0<0<x_1 \Leftrightarrow f(x)\lesseqgtr f(0) \quad \text{for} \quad x\lesseqgtr 0. \tag{24}$$

Applying this to (23) gives

$$\sum_j \frac{\operatorname{sgn}(y_j)}{u(y_j)}\left[\frac{u(x+y_j)}{u(x)}-\frac{u'(y_j)}{u'(0)}\right]\lesseqgtr 0 \quad \text{for all} \quad x\gtreqless 0. \tag{25}$$

Applying (24) with y_j in place of x_j now gives

$$\frac{u'(x)}{u(x)}+\frac{u'(y)}{u(y)}\lesseqgtr\frac{u(x+y)u'(0)}{u(x)u(y)}+\frac{u''(0)}{u'(0)} \quad \text{whenever} \quad xy\lesseqgtr 0. \tag{26}$$

Multiplying through by $u(x)u(y)$ gives (21). *Q.E.D.*

6. LOCAL ANALYTICAL CONDITIONS

For concavity (risk aversion) and decreasing risk aversion, the necessary local conditions $u''\leqslant 0$ and $r'\leqslant 0$ derived by considering small gambles turn out to be globally sufficient as well. This proves not to be true for proper risk aversion, however, as shown below by a clever example due to John Lindsey.

[15] Quadratic utility provides an example, since it satisfies (4), as mentioned earlier, and (4) implies (21). Alternatively, for $u(w)=-w^2$, $w<0$, the left-hand side minus the right-hand side of (21) is $-(w-x)^2(w-y)^2/w$ by direct calculation.

To derive local conditions, we first let $y \to w$ in (21), obtaining from the second-order terms (lower terms vanish)

$$[u(x)-u(w)]r'(w) \leqslant u'(x)[r(x)-r(w)] \quad \text{for all } w \text{ and } x. \tag{27}$$

This can of course be written in many algebraically equivalent forms, since $r(w) = -u''(w)/u'(w)$, but it is not fully local in that $x \neq w$. Letting $x \to w$ now gives, again from second-order terms,

$$r''(w) \geqslant r'(w)r(w) \quad \text{for all } w. \tag{28}$$

Note that on the right-hand side of (28), $r \geqslant 0$ and $r' \leqslant 0$. If $r'(w) = 0$, then $r(x)$ is constant for $x \geqslant w$, since (28) is equivalent to $u'r'$ increasing while $r' \leqslant 0$ implies $u'r' \leqslant 0$. On the interval where $r' < 0$, r can be partly concave and partly convex, but (28) limits how convex it can be.

Although (27) and (28) are necessary conditions derived by taking one or both gambles small in the definition of fixed-wealth proper, the example below shows that (28), even with strict inequality, does not imply (27) or, therefore, properness. Precise definitions of local properties for which (27) and (28) are sufficient as well as necessary conditions would be fussy and tricky to give, and the game not worth the candle. We content ourselves with the following statement of necessity.

THEOREM 5: *Fixed-wealth proper risk aversion implies* (27) *even if one gamble in the defining condition* (2) *is required to be arbitrarily small, say* $P(|\tilde{y}| \leqslant \varepsilon) = 1$ *for some* $\varepsilon > 0$, *and it implies* (28) *even if both gambles are required to be arbitrarily small.*

PROOF: If $|y_i| \leqslant \varepsilon$ is required in the proof of Theorem 4, (21) still follows for $|y - w| < \varepsilon$, and similarly for $\tilde{x}$. Conditions (27) and (28) follow easily by Taylor expansion. *Q.E.D.*

EXAMPLE 3 (John Lindsey): If $u(x) = x + \sinh x = x + \frac{1}{2}e^{x} - \frac{1}{2}e^{-x}$, $x \leqslant 0$, then $u'(x) = 1 + \cosh x$, $r(x) = -\sinh x/(1+\cosh x) = -\tanh(x/2)$, $r' = -1/u'$, $r'' = u''/u'^2 = r'r$, and equality holds in (28). However, (27) becomes

$$u(w)-u(x) \leqslant u'(w)u'(x)[r(x)-r(w)] \quad \text{for all } w \text{ and } x, \tag{29}$$

and since interchanging w and x merely reverses the sign on both sides, this further implies that equality holds, which is false. Thus (28) holds, but (27) and therefore (21) do not and u is not proper. To make the inequality in (28) strict, let $r_1(x) = r(ax)$, $a > 1$; then $r_1''(x) = a^2 r''(ax) > ar'(ax)r(ax) = r_1'(x)r_1(x)$. Wherever (27) fails for r it will also fail for r_1 if a is close enough to 1, by continuity. To extend r_1 to $x > 0$, find a w_0 such that (27) fails for some x, $w < w_0$. On a small interval (w_0, w_1) increase r_1'' in such a way that (28) holds with strict inequality, $r_1' < 0$, $r_1 > 0$, and $r_1''(w_1) > 0$. On (w_1, ∞), replace r_1 by a positive, continuous, strictly decreasing, strictly convex function without changing its value or derivatives at w_1. Then (28) holds with strict inequality on $(-\infty, \infty)$ but (27) fails. The failure can be brought to a point of positive wealth by a change of origin.

Proper risk aversion is a descriptive property of a utility function with important implications for behavior. If its normative appeal is accepted, it should be useful in refining the strategy that began by developing von Neumann-Morgenstern utility from axiomatic restrictions on choice procedures. Agree to a set of normative principles that restrict the available class of utility functions; then select your actual function by calibrating and fitting one or a few parameters. The thrust of this suggestion may appear to run counter to a major theme in recent work in decision analysis, chronic violations of the traditional axioms. These two strands of research, however, are actually mutually reinforcing. Individuals have demonstrated in a great variety of contexts and arenas that as intuitive beings they are poor at making consistent decisons, especially where uncertainties are involved. We should hardly expect them to be good at defining preferences directly, whether in laboratory or real-world settings. There would seem to be three options: throw up our hands in despair, seek a satisfactory replacement for the well-developed theory of decision making under uncertainty, or attempt to develop additional tools—including normatively imposed constraints—that might simplify the task of defining preferences. The concept of proper risk aversion contributes to the third approach. Fortunately, a wide class of analytically tractable functions is proper.

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