Econometrica, Vol. 42, No. 4 (July, 1974)

THE TOPOLOGY OF PARETO-OPTIMAL REGIONS WITH PUBLIC GOODS

BY RICHARD J. ZECKHAUSER AND MILTON C. WEINSTEIN[1]

This analysis generalizes the concept of a private-goods contract curve to an economy with public goods. It provides a derivation of the properties of the set of Pareto-optimal points in n-space. The paper concentrates on situations where the taxation system for financing the public goods is externally imposed; the efficiency goal is thus "mechanism-constrained Pareto optimality." The results of the main analysis are applied to assess performance of several group decision procedures.

1. INTRODUCTION

THE EDGEWORTH BOX is the comfortable womb of us microeconomists. Raise the subject of Pareto-optimal points, and "contract curve" is our primordial thought. Our mind treads confidently on the familiar terrain of a two-man world with two private goods. In this paper we venture out a step or two to consider the shapes and properties of Pareto-optimal regions when goods may be public and consumers may be many.

The major task of this paper is to develop a public-goods analogue to the market-exchange model's concept of the contract curve, i.e., a description of the locus of Pareto-optimal points in a world with public goods. Then we turn briefly to a discussion of alternative procedures for attempting to secure outcomes within this region.

2. FACTORING PRIVATE GOODS OUT OF THE ANALYSIS

Exposition is simplest if we tuck private goods neatly aside before turning to public goods, our topic of primary interest.[2] This is most easily done by

[1] The authors would like to thank Professor Kenneth Arrow and a referee for their most helpful comments. This research was supported by National Science Foundation grants GS-2862x and GR-58.

[2] The contract curve concept is readily generalized to a private-goods-only world with many goods and many consumers. To represent the endowments of n individuals in each of m goods, it is necessary to employ mn dimensions. However every time we constrain ourselves by limiting the total of all individuals' endowments on a good we can save one dimension; one individual's endowment is merely the balance when the allocations of the others are subtracted from the total. The Edgeworth box places such constraints on two goods; hence it must be of $2 \times (2 - 1) = 2$ dimensions. In the many-many world, if the total endowments of all goods are known, we shall need $m(n - 1)$ dimensions to represent all possible allocations.

What then of the dimensionality of the locus of efficient points? With two individuals and two goods we get the one-dimensional contract curve. Add an extra good, and the Edgeworth box goes three dimensional; but a moment's thought about "nested shells" reveals that the locus of efficient points remains a one-dimensional figure; it is a curve in three-space. This result generalizes. With two individuals, the locus of efficient points is a one-dimensional figure: topologically speaking, a one-simplex.

With extra individuals, unfortunately, our powers of spatial conceptualization are likely to be inadequate. We might conjecture correctly the following theorem.

THEOREM A: *With n individuals in a world of private goods and strictly convex preferences, the locus of efficient points will be an $(n - 1)$-dimensional figure.*

We shall withhold our sketch of the proof of this theorem to footnote 9, after we have developed more machinery in the context of the public goods example.

incorporating the financial mechanism for providing the public-goods bundle into the analysis. For each such bundle, there will be an associated market equilibrium in private goods. At this equilibrium, each individual will receive his particular allocation of private goods. Given the functional relationship between the public-goods bundle and his associated private-goods allocation, every individual can attach a utility value to each possible public-goods bundle.[3] In what follows, we shall deal with derived utility functions of this nature, where private-goods arguments have been eliminated and the preferences for public goods include consideration of the mechanism through which they are provided.

3. MECHANISM-CONSTRAINED PARETO OPTIMALITY

To restrict attention to particular procedures through which public goods can be financed makes our analysis more relevant to real world considerations. (Financing mechanisms, unlike many other variables economists consider, are not parameters that can be readily varied.) However, it must be understood that the points that are efficient, given this restriction, are merely constrained maxima within limited sets. Does it make sense to label such points Pareto optimal? We believe so. The term Pareto optimality has taken on such extraordinary significance in contemporary discussion, that bundles of goods that are to be thought of as desirable or efficient must retain a significant measure of the approbation conveyed by this term. Still, it seems appropriate to add a modifier that indicates the nature of the restriction on the feasible set. Hence, we propose that outside the confines of this paper the term "mechanism-constrained Pareto optimality" be applied to constrained optima when the institutional financing mechanism is externally imposed.[4]

We could conduct our analysis at a greater level of generality by banishing mechanism constraints and treating private goods as endogenous variables that are manipulable arguments of the optimization process. This, the classic Samuelson approach, specifies that the efficient level of provision for a public good creates a balance between individuals' summed marginal willingness to pay for the good

[3] In some cases, the working of the market mechanism will be extraordinarily simple. For example, private goods prices will be unaffected if the public goods market is small relative to the relevant private goods market (as it might be for a small country or local community), or if private goods have one-factor, constant-returns technologies. In these instances individuals will be able to calculate their utilities in straightforward fashion once they know the tax scheme.

In other instances, simulations of complicated market processes will be required before individuals can calculate their welfares associated with particular public goods bundles. Here, as whenever individuals are concerned with matters of social choice, they must calculate outcomes as best they can. Fortunately, as far as computational simplicity is concerned, real world public-goods allocations are frequently made on an incremental basis.

[4] To employ instead the term "second best" would seem to lend less clarity. Following on the lead of the pioneering article [5], we would urge that second-best optimization be accorded a restricted domain of reference. It should be applied to discussions of sets of marginal efficiency conditions when certain other sets are not subject to influence. The Lipsey-Lancaster result demonstrates that "if there is introduced into a general equilibrium system a constraint which prevents attainment of one of the Paretian conditions, the other Paretian conditions, although still attainable, are in general, no longer desirable" [5, p. 156]. Any broader philosophical interpretation of their important result would distort and weaken those authors' intent.

and the good's marginal cost of provision (see Samuelson [7 and 8]). This is a maximization process without institutional constraints detailing who pays for the inframarginal units.[5] We believe our major conclusions would carry through in this unconstrained framework, but since a multiplicity of complex corner conditions come into play, we have not been able to rigorize our intuition on this matter. It should be noted that our analysis turns out to be perfectly general if all private goods generate externalities to all individuals, for then all goods would be public, and our model would address the entire world.

4. THE STRUCTURE OF PREFERENCES FOR PUBLIC GOODS WITH IMPLICIT ACCOUNT OF THEIR FINANCE MECHANISM

With financing considerations included in the evaluation of public-goods bundles, there will be levels of provision beyond which more is worse, not better. We shall assume that an individual's preferences for public goods have the form shown in Figure 1. He has an interior bliss point, and his indifference contours are closed convex curves. (He prefers a weighted average of two public-goods bundles to the lottery on the extremes that employs the same weights as probabilities.)

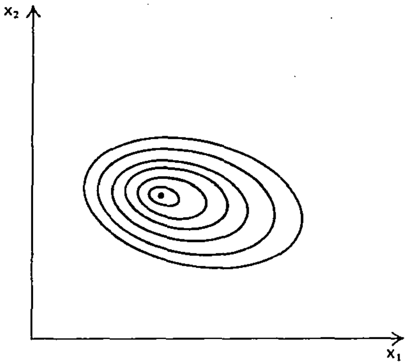

FIGURE 1.—A representative individual's preferences for public goods.

A variety of sets of circumstances may lead to preference structures of this nature. Perhaps the set that is most comprehensible and representative occurs when (i) individuals have strictly convex preferences for private and public goods, (ii) there are no regions of increasing returns to scale in public-goods provision, (iii) an individual's cost share in providing a public good is constant at the margin, and (iv) public-goods provision does not appreciably alter private-goods prices.

In particular, consider the following model. Represent private goods by w's and public goods by x's. An individual's utility for a private and public goods bundle (x, w) can be represented by $s(x, w)$, a function assumed to be strictly

[5] See Aaron and McGuire [1] for discussion of distributional characteristics of public goods optima, when provision is constrained by a finance mechanism.

concave.[6] For every budget constraint B, price vector for private goods p, and public-goods vector x, he will select $w = w^*$ to maximize $s(x^*, w)$, subject to $p'w \leqslant B$. This constrained maximization process yields a derivative utility function

$$(1) \qquad t(B, x) = \max_w s(x, w) \text{ subject to } p'w \leqslant B.$$

One further step is required if we are to suppress private goods completely; we must allocate the individual cost shares through which public goods are financed. Say our representative individual pays $c(x)$ towards public goods bundle x. Take as an initial position bundle x_0, and give our individual a budget B_0. For another bundle x', his budget will be $B' = B_0 + c(x_0) - c(x')$. However, with B_0 and x_0 given parameters, we can write

$$(2) \qquad u(x) = t(B_0 + c(x_0) - c(x), x).$$

This function u provides us with the desired utility function for public goods private goods as such have been whisked off into parameterization.

4.1. *The Properties of the Utility Function for Public Goods*

The function $u(x)$ represents at one time the valuation of public goods and the sacrifices entailed in providing them. To this extent, it might represent a typical taxpayer's preferences. The taxpayer's bliss point is achieved at an internal location where the marginal cost to him of providing more of each public good just equals his valuation of a marginal unit.

One final assumption enables us to tighten the final screw on our model. We assume that (at least at the margin) each public good is provided on a constant cost-sharing basis. If an individual pays δ_j towards an additional dollar of expenditure on public good j, he will pay $2\delta_j$ if two dollars more are spent. Represent the individual's vector of shares as $\delta = (\delta_i)$. His total cost share is then of the form $c(x) = K + \delta' x$.

Together these assumptions imply that if the indifference curves of $s(x, w)$ are strictly convex (i.e., if $s(x, w)$ has a strictly concave cardinal representation), then $u(x)$ will have strictly convex iso-utility contours as in Figure 1.[7] In what follows we shall be concerned with public-goods cases where preference contours are of this shape.

[6] The w_i are expressed in dollars spent by the individual, and the x_i are expressed in total dollars spent on the public good. It is an easy matter to show that if (i) preferences are concave with the arguments expressed in some units other than dollars, and (ii) there are nondecreasing costs in providing all goods when expressed in those units, then preferences will be concave when the arguments are expressed in dollars. We use both of these assumptions at this point.

[7] This can be easily demonstrated as follows. First, we show that if $s(x, w)$ is strictly concave, then $t(B, x)$ as defined before is also strictly concave. We will show that for $0 < \lambda < 1$, then for any (B_1, x_1) and (B_2, x_2),

$$(\text{i}) \qquad t(\lambda B_1 + (1 - \lambda)B_2, \lambda x_1 + (1 - \lambda)x_2) > \lambda t(B_1, x_1) + (1 - \lambda)t(B_2, x_2).$$

Now by definition of the function t,

$$t(\lambda B_1 + (1 - \lambda)B_2, \lambda x_1 + (1 - \lambda)x_2) = \max_w s(\lambda x_1 + (1 - \lambda)x_2, w)$$

subject to

$$p'w \leqslant \lambda B_1 + (1-\lambda)B_2$$

By defining dummy variables w_1 and w_2 such that $\lambda w_1 + (1-\lambda)w_2 = w$, we get the desired result by the following chain of equalities and inequalities:

$$t(\lambda B_1 + (1-\lambda)B_2, \lambda x_1 + (1-\lambda)x_2) = \max_w s(\lambda x_1 + (1-\lambda)x_2, w) \qquad \text{(by definition of } t\text{)}$$

subject to

$$p'w \leqslant \lambda B_1 + (1-\lambda)B_2$$

$$= \max_{w_1, w_2} s(\lambda x_1 + (1-\lambda)x_2, \lambda w_1 + (1-\lambda)w_2)$$

subject to

$$\lambda p'w_1 + (1-\lambda)p'w_2 = \lambda B_1 + (1-\lambda)B_2 \qquad \text{(by introducing dummy variables)}$$

$$> \max_{w_1, w_2} \lambda s(x_1, w_1) + (1-\lambda)s(x_2, w_2)$$

subject to

$$\lambda p'w_1 + (1-\lambda)p'w_2 = \lambda B_1 + (1-\lambda)B_2 \qquad \text{(by strict concavity of } s\text{)}$$

$$\geqslant \max_{w_1, w_2} \lambda s(x_1, w_1) + (1-\lambda)s(x_2, w_2)$$

subject to

$$p'w_1 \leqslant B_1,$$

$$p'w_2 \leqslant B_2 \qquad \text{(because the constraints have been made stricter)}$$

$$= \max_{w_1} \lambda s(x_1, w_1)$$

subject to

$$p'w_1 \leqslant B_1$$

$$+ \max_{w_2} (1-\lambda)s(x_2, w_2)$$

subject to

$$p'w_2 \leqslant B_2 \qquad \text{(by the principle of decomposition)}$$

$$= \lambda t(B_1, x_1) + (1-\lambda)t(B_2, x_2) \qquad \text{(by definition of } t\text{),}$$

which completes the derivation of (i).

Now we complete the proof by showing that if $t(B_1, x_1)$ is strictly concave, then $u(x)$ as defined before by a fixed-proportion tax scheme is also strictly concave. By definition (2),

$$u(\lambda x_1 + (1-\lambda)x_2) = t(B_0 + c(x_0) - c[\lambda x_1 + (1-\lambda)x_2], \lambda x_1 + (1-\lambda)x_2).$$

Now, by assumption, $c(x) = K + \delta' x$, so that

$$\begin{aligned} u(\lambda x_1 + (1-\lambda)x_2) &= t(B_0 + \delta' x_0 - \lambda\delta' x_1 - (1-\lambda)\delta' x_2, \lambda x_1 + (1-\lambda)x_2) \\ &= t(\lambda(B_0 + \delta' x_0 - \delta' x_1) + (1-\lambda)(B_0 + \delta' x_0 - \delta' x_2), \\ &\qquad \lambda x_1 + (1-\lambda)x_2). \end{aligned}$$

By the strict concavity of t, and by reapplying the definition of u, we get the result that

(ii) $$u(\lambda x_1 + (1-\lambda)x_2) > \lambda u(x_1) + (1-\lambda)u(x_2).$$

This completes the proof that the strict concavity of $s(x, w)$ implies the strict concavity of $u(x)$. Thus, we have shown that under reasonable assumptions, Figure 1 displays the preference structure for public goods with which we are concerned.

We believe a slightly relaxed variant of our major result holds if preferences are merely convex, not strictly convex. However, the special cases multiply manyfold, and some powerful analytic procedures no longer apply. Together, these factors have prevented us from proving our conjecture.

5. PUBLIC GOODS WHERE RESOURCE COST IS NOT A PRIMARY ELEMENT OF INTEREST

It should be recognized that this analysis pertains to a much more general class of social choice processes than those normally considered under the heading "procedures for public goods provision." Most of conventional public goods theory is concerned with goods whose production requires substantial resources: national defense, highways, recreation areas. These goods can usually be tallied up in some form of physical units. Provision costs aside, more of them is always better.

There is an entire class of things that share many of the characteristics of public goods, but not the properties just mentioned. We think here of policy variables that for the most part represent non-tangible goods, such things as a foreign policy or government attitudes toward law and order. These variables, like other public goods, enter the utility functions of all individuals. Furthermore, the way in which they enter is not related to who selects or provides them.

They differ from more traditional public goods in that (i) they do not lend themselves to ready physical measures, so that scales indicating their values are somewhat arbitrary, (ii) the cost of providing them is a relatively unimportant consideration in selecting their optimal levels, and (iii) independent of costs of provision, more of them is not always better.

To analyze these goods in our framework it is necessary to have some underlying ordinal scale on which they can be measured. (A number of scales may be used if it is desired to factor goods into many dimensions.) Then with the possible aid of monotonic transformations of these scales, it must be possible to represent all individual's preferences for these goods in the form shown in Figure 1. Given this, our discussion allows for, indeed welcomes, consideration of this frequently slighted class of public goods.

The possible inclusion of this additional class of public goods gives further justification for our move away from the classic approach to efficient provision which relates marginal rates of substitution of money for the good and marginal costs of provision. These concepts can only be given a most strained interpretation in this context. Our replacement notion, a region of Pareto optimality, on the other hand, has a natural straightforward interpretation for these goods.

6. THE MAJOR TASK

These excursions have hopefully defined the domain of our major task, which is to characterize the structure of the Pareto-optimal region when individuals' preferences for public goods have the form illustrated in Figure 1.

6.1. *The Case of Two Goods and Three Individuals*

The simplest case of interest has two goods and three individuals. Denote the goods by x_1 and x_2, and call the individuals A, B, and C. Each individual is assumed to have ordinal preferences which can be represented by a function for strictly concave expenditures on these two goods, and in the context of public goods described above, this function will be assumed to have a (global) maximum at some bliss point. Denote these preference functions by $u_1(x)$, $u_2(x)$, and $u_3(x)$, where $x = (x_1, x_2)$. Label the respective bliss points (i.e., the points where $\nabla u_1 = 0$, $\nabla u_2 = 0$, and $\nabla u_3 = 0$, respectively)[8] α, β, and γ. Figure 2 shows a typical set of indifference contours in (x_1, x_2) space. Note that the assumption of strict concavity is sufficient to assume that the indifference contours are strictly convex. We shall also assume that the preference functions are continuously differentiable.

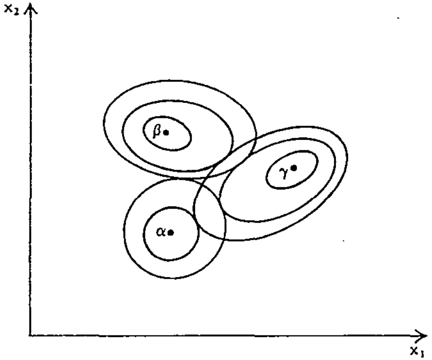

FIGURE 2.—Typical indifference contours for the three-person, two-good case.

Our principal concern is to identify the locus of Pareto-optimal points. Clearly, the locus of tangencies between the indifference contours of A and B are Pareto-optimal: at any such point neither A nor B can be made better off without making the other worse off. Similarly, the locus of tangencies between the indifference contours of A and C are Pareto-optimal; likewise for B and C. Figures 3a and 3b show a typical set of two-individual "contract curves" thus defined. The result to be proved is that the set of Pareto-optimal points is precisely the closure of the "triangular" region bounded by the three contract curves, precisely the shaded region in Figure 3b.

Unfortunately, the contract curves need not appear as they do in Figure 3a. Indeed, the two-person contract curves may cross each other in a variety of ways. Still, it will be shown in all cases that the Pareto-optimal set is the region bounded by the contract curves. Some possibilities are displayed in Figures 4a–4f. A general characterization of the possible configurations is one of the results to be derived formally below. It is illuminating, however, to examine this characterization briefly before moving on to the formal statement and proof of the basic theorem.

[8] We assume throughout this analysis that boundary optima are ruled out, i.e., that the relevant range of preference is well removed from the coordinate axes.

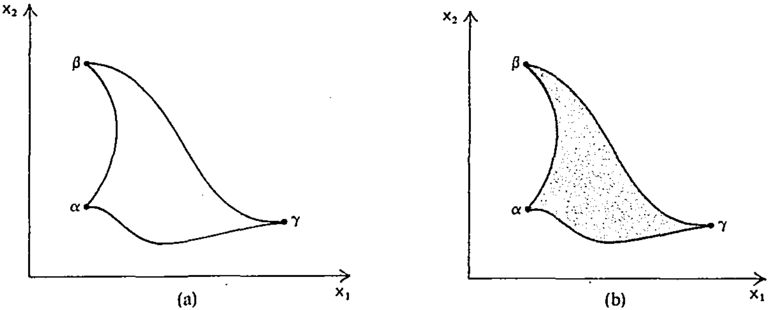

FIGURE 3a.—Typical two-person contract curve for the three-person, two-good case
FIGURE 3b.—Typical Pareto-optimal region for the three-person, two-good case

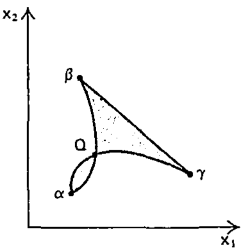

FIGURE 4a

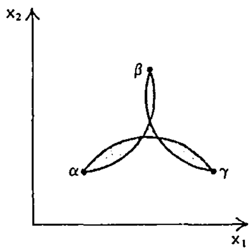

FIGURE 4b

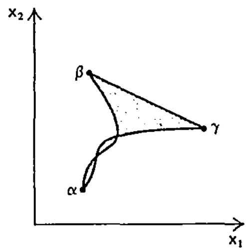

FIGURE 4c

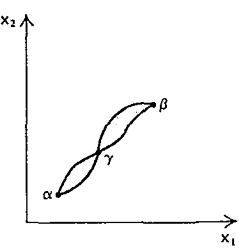

FIGURE 4d

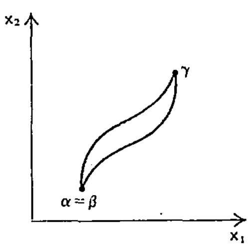

FIGURE 4e.

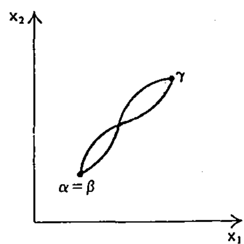

FIGURE 4f

FIGURES 4a–4f.—A general characterization of the possible situations.

Under the differentiability and convexity assumption we have made, the Pareto-optimal set is precisely the set of points x which maximize

$$\lambda_1 u_1(x) + \lambda_2 u_2(x) + \lambda_3 u_3(x) \tag{3}$$

for some triplet of the set $\{(\lambda_1, \lambda_2, \lambda_3): \lambda_1 + \lambda_2 + \lambda_3 = 1, \lambda_1 \geqq 0, \lambda_2 \geqq 0, \lambda_3 \geqq 0\}$. The set of λ-triplets corresponds to the standard 2-simplex S_2, the triangular region drawn in Figure 5. Thus, the above maximization defines a mapping $\phi: S_2 \to R^2$.

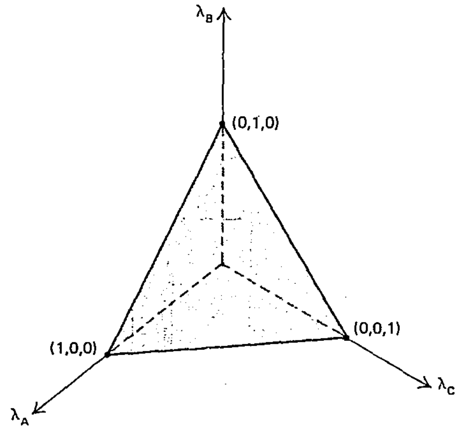

FIGURE 5.—Standard 2-simplex.

It will be shown that the image of S_2 under ϕ is either topologically homeomorphic to the 2-simplex S_2 (the case in Figure 3b), or else it is topologically homeomorphic to a set generated by "collapsing" S_2 continuously along one or more (possibly uncountably many) line segments, which must generally be non-intersecting. Figures 4a–4f provide illustrative examples. The region in Figure 4a is homeomorphic to S_2, collapsed along the line segment connecting some $(\lambda_1, \lambda_2, 0)$ with some $(\lambda_1', 0, \lambda_3')$. The region in Figure 4c is homeomorphic to S_2, collapsed along the line segments connecting two such pairs of λ's, provided that those segments do not intersect. The region in Figure 4d is obtained by collapsing S_2 along a line segment connecting $(0, 0, 1)$ with some $(\lambda_1, \lambda_2, 0)$. The region in Figure 4e is obtained by collapsing S_2 along the line segment between $(1, 0, 0)$ and $(0, 1, 0)$. An infinity of topological variations is possible, although it should be noted that the situations shown in Figures 6a, 6b, and 6c are *not* possible.

To state these ideas formally, we shall employ the notion of a topological *quotient space*, which is defined as follows. Let R be an equivalence relation on S_2, which defines a partition of the set S_2. The quotient space S_2/R is the set of all subsets in the partition, together with the (minimal) topology which makes the point-to-set projection map $\pi_R: S_2 \to S_2/R$ continuous. We may think of a quotient

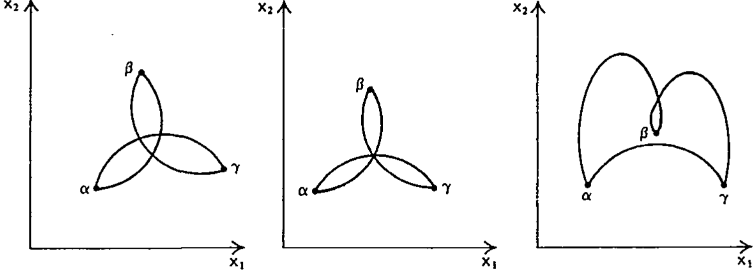

FIGURE 6.—"Impossible" situations.

space simply as a set of points—in this case the region S_2 as shown in Figure 5—"collapsed" in some specified manner. Thus, if R is simply the relation which associates each point in S_2 with itself alone, then S_2/R is essentially S_2, and we are in the case of Figure 3b. If R is a relation which associates points along a line segment with each other, but which associates all other points with themselves alone, then S_2/R is topologically equivalent to the region in Figure 4a. This region is essentially S_2, "collapsed" along a line segment connecting two edge points. If R were the universal equivalence relation, then S_2/R would collapse to a single point and α, β, and γ would coincide; this would be the case of no conflict of interest. What we wish to show is that the Pareto-optimal set is always homeomorphic to S_2/R, where R collapses points along an arbitrary number of line segments connecting boundary points of S_2. These line segments, furthermore, are always non-intersecting except in the degenerate case where there is no conflict of interest.

6.2. *The Course to be Followed*

Our immediate task is to develop a proof of these propositions. Three lemmas provide the stepping stones to our objective as stated and proved in Theorem 1. This result will then be generalized to the case of m public goods and n individuals. Theorem 3 says that, in higher dimensions of goods, the Pareto-optimal region in m-space is essentially the region bounded by the $(m-1)$-dimensional "contract surfaces" formed by considering the individuals m at a time (e.g., a "ball" in three-space). We turn first to the proof of Theorem 1.

LEMMA 1: *Let $u_i(x)$ $(i = 1, 2, 3)$ be strictly concave, continuously differentiable, real-valued functions on R^2. Let $\lambda \in S_2$. Define a function $\phi : S_2 \to R^2 (\phi : \lambda \mapsto x_\lambda)$ by*

$$\max_x \sum_{i=1}^{3} \lambda_i u_i(x) = \sum_{i=1}^{3} \lambda_i u_i(x_\lambda).$$

Then ϕ is well defined and continuous.

PROOF: Since the u_i are strictly concave, so is $\Sigma_{i=1}^3 \lambda_i u_i$. Therefore, the maximand x_λ is unique, so that ϕ is well defined. Furthermore, since $\Sigma_{i=1}^3 \lambda_i u_i$ is a continuous function of λ, the maximand x_λ varies continuously with λ, by well known results in convex analysis (see, for example, Lancaster [3, p. 349]). Thus, ϕ is well defined and continuous.

LEMMA 2: *If $x = \phi(\lambda) = \phi(\lambda')$ for $\lambda, \lambda' \in S_2$, then either (i) $\lambda = \lambda'$, or (ii) exactly two of the following hold:*

1. $\phi(\lambda) = \phi(\lambda^*)$ *for some* λ^* *with* $\lambda_3^* = 0$;
2. $\phi(\lambda) = \phi(\lambda^{**})$ *for some* λ^{**} *with* $\lambda_1^{**} = 0$;
3. $\phi(\lambda) = \phi(\lambda^{***})$ *for some* λ^{***} *with* $\lambda_2^{***} = 0$;

furthermore,

$$\begin{aligned}\phi^{-1}(\phi(\lambda)) &= \{\mu\lambda^* + (1-\mu)\lambda^{**} : 0 \leqslant \mu \leqslant 1\} \quad \textit{if (1) and (2) hold,}\\ &= \{\mu\lambda^* + (1-\mu)\lambda^{***} : 0 \leqslant \mu \leqslant 1\} \quad \textit{if (1) and (3) hold,}\\ &= \{\mu\lambda^{**} + (1-\mu)\lambda^{***} : 0 \leqslant \mu \leqslant 1\} \quad \textit{if (2) and (3) hold;}\end{aligned}$$

or (iii) all of 1, 2, *and* 3 *hold, and either*

(a) $\lambda^* = \lambda^{***} = (1,0,0)$ *and* $\phi^{-1}(\phi(\lambda)) = \{\mu\lambda^* + (1-\mu)\lambda^{**} : 0 \leqslant \mu \leqslant 1\}$,

(b) $\lambda^* = \lambda^{**} = (0,1,0)$ *and* $\phi^{-1}(\phi(\lambda)) = \{\mu\lambda^* + (1-\mu)\lambda^{**} : 0 \leqslant \mu \leqslant 1\}$,

or

(c) $\lambda^{**} = \lambda^{***} = (0,0,1)$ *and* $\phi^{-1}(\phi(\lambda)) = \{\mu\lambda^{**} + (1-\mu)\lambda^{***} : 0 \leqslant \mu \leqslant 1\}$;

or (iv) all of 1, 2, *and* 3 *hold, and*

$$\lambda^* = (1,0,0), \quad \lambda^{**} = (0,1,0), \quad \lambda^{***} = (0,0,1),$$

and

$$\phi^{-1}(\phi(\lambda)) = S_2.$$

Equivalently, if x is the Pareto-optimal point corresponding to two sets of weights, then either (i) the weights are the same; or (ii) x lies on exactly two two-person contract curves (point Q in Figure 4a), and corresponds precisely to a line segment of λ's in S_2; or (iii) x lies on all three contract curves, in which case x must be a bliss point for at least one individual (point γ in Figure 4d), or possibly two individuals (point α in Figure 4e) (as in case (ii), x corresponds to a line segment of λ's in S_2); or (iv) all three bliss points coincide at x and there is no conflict of interest.

PROOF: Let $\phi(\lambda_1, \lambda_2, \lambda_3) = \phi(\lambda'_1, \lambda'_2, \lambda'_3) = (x_1, x_2)$. Denote the gradient vectors of $u_1(x)$, $u_2(x)$, and $u_3(x)$ (evaluated at x) by ∇_1, ∇_2, and ∇_3. Then

$$\lambda_1\nabla_1 + \lambda_2\nabla_2 + \lambda_3\nabla_3 = \lambda'_1\nabla_1 + \lambda'_2\nabla_2 + \lambda'_3\nabla_3 = 0 \tag{4}$$

at x.

We consider the cases where x lies on exactly zero, one, two, and three contract curves, and show that the only possibilities are (i) through (iv) above.

Suppose that x does not lie on any contract curve. Therefore, neither $\lambda_1, \lambda_2, \lambda_3, \lambda'_1, \lambda'_2$, nor λ'_3 can be zero. Now, from (4),

$$\lambda_2\nabla_2 + \lambda_3\nabla_3 = -\lambda_1\nabla_1 \tag{5a}$$

and

$$\lambda'_2\nabla_2 + \lambda'_3\nabla_3 = -\lambda'_1\nabla_1, \tag{5b}$$

so that

$$\lambda_2\nabla_2 + \lambda_3\nabla_3 = \frac{\lambda_1}{\lambda'_1}(\lambda'_2\nabla_2 + \lambda'_3\nabla_3) \tag{6}$$

Since x is not on a contract curve, neither ∇_2 nor ∇_3 can be zero, nor can they be collinear. (If ∇_2 and ∇_3 were collinear with $\nabla_2 = k\nabla_3$, then if $k < 0$, x is on the contract curve $\beta\gamma$; if $k > 0$, (5a) and (5b) show that x is on the contract curves $\alpha\beta$ and $\alpha\gamma$.) Thus, ∇_2 and ∇_3 form a basis for R^2, and the representations in (6) must be the same. Therefore,

$$\frac{\lambda_1}{\lambda'_1} = \frac{\lambda_2}{\lambda'_2} = \frac{\lambda_3}{\lambda'_3},$$

and since $\Sigma\lambda_i = \Sigma\lambda'_i = 1$, we get the result that $\lambda_1 = \lambda'_1, \lambda_2 = \lambda'_2$, and $\lambda_3 = \lambda'_3$. Thus, if x is on no contract curve, then $\lambda = \lambda'$ and case (i) must hold.

Now suppose that x lies on exactly one contract curve, say $\alpha\beta$. We can therefore take $\lambda_3 = 0$ without loss of generality. Furthermore, both λ_1 and λ_2 are nonzero, since a bliss point automatically lies on two contract curves (which is ruled out here). Let h be any vector orthogonal to ∇_1, i.e., such that $\nabla_1 h = 0$. Then, since $\nabla_2 = (-\lambda_1/\lambda_2)\nabla_1$, $\nabla_2 h = 0$. Thus, by (4),

$$(\lambda'_1\nabla_1 + \lambda'_2\nabla_2 + \lambda'_3\nabla_3)h = 0,$$

so that

$$\lambda'_3\nabla_3 h = 0.$$

Therefore, either $\lambda'_3 = 0$, $\nabla_3 = 0$, or ∇_3 is collinear with ∇_1 and ∇_2. If $\lambda'_3 = 0$, then $\lambda_1 = \lambda'_1$ and $\lambda_2 = \lambda'_2$, and case (i) holds. If $\nabla_3 = 0$, then x is a bliss point which is ruled out. If ∇_3 is collinear with ∇_1 and ∇_2, then x is on either $\alpha\gamma$ or $\beta\gamma$ in addition to $\alpha\beta$, which is ruled out here. Thus, if x lies on exactly one contract curve, then case (i) must hold: $\phi^{-1}(x)$ is unique.

Thus far, we have shown that if $\phi^{-1}(x)$ is not a single point, then x must lie on at least two contract curves. The next case to consider is where x lies on exactly two contract curves. It is claimed that either the conditions of case (ii) must hold, or else x is the bliss point for one of the individuals.

Suppose x lies on $\alpha\beta$ and $\alpha\gamma$, but not on $\beta\gamma$. We might as well let $\lambda = (\lambda_1, \lambda_2, 0)$ and $\lambda' = (\lambda'_1, 0, \lambda'_3)$. If λ_2 or λ'_3 is zero, then $x = \alpha$. If either λ_2 or λ'_3 is zero, then they both must be zero: otherwise either $x = \alpha = \beta$ or $x = \alpha = \gamma$, but these are ruled out because x would then be on all three contract curves. Therefore, if either λ_2 or λ'_3 is zero, then they are both zero, and case (i) holds. Now suppose $\lambda_2 \neq 0$, $\lambda'_3 \neq 0$. Let λ'' be such that $\phi(\lambda'') = x$. We claim that $\lambda'' = \mu\lambda + (1 - \mu)\lambda'$ for some μ between zero and one.

Consider first the case where $\lambda_1 \neq \lambda'_1$. For now we can at least write $\lambda''_1 = \mu\lambda_1 + (1 - \mu)\lambda'_1$, but we cannot yet assert that $0 \leqslant \mu \leqslant 1$. Now,

(7a) $$\lambda_1\nabla_1 + \lambda_2\nabla_2 = 0$$

and

(7b) $$\lambda'_1\nabla_1 + \lambda'_3\nabla_3 = 0.$$

Thus,

(8) $$\nabla_3 = \pi\nabla_2,$$

where

(9) $$\pi = (\lambda'_1\lambda_2)/(\lambda_1\lambda'_3).$$

By (7a) and (7b),

$$(\mu\lambda_1 + (1 - \mu)\lambda'_1)\nabla_1 + \mu\lambda_2\nabla_2 + (1 - \mu)\lambda'_3\nabla_3 = 0,$$

or, equivalently,

(10) $$\lambda''_1\nabla_1 + \mu\lambda_2\nabla_2 + (1 - \mu)\lambda'_3\nabla_3 = 0.$$

However,

(11) $$\lambda''_1\nabla_1 + \lambda''_2\nabla_2 + \lambda''_3\nabla_3 = 0.$$

Combining (9), (10), and (11), and observing that $\nabla_2 \neq 0$, we get

(12) $$\mu\lambda_2 + (1 - \mu)\lambda'_3\pi = \lambda''_2 + \lambda''_3\pi.$$

But also,

$$\begin{aligned} \lambda''_2 + \lambda''_3 &= 1 - \mu\lambda_1 - (1 - \mu)\lambda'_1 \\ &= 1 - \lambda'_1 - \mu(1 - \lambda'_1) + \mu(1 - \lambda_1) \\ &= (1 - \mu)(1 - \lambda'_1) + \mu(1 - \lambda_1) \end{aligned}$$

(13) $$= \mu\lambda_2 + (1 - \mu)\lambda'_3.$$

Subtracting (13) from (12),

$$(14)\qquad \lambda_3''(1-\pi) = (1-\mu)\lambda_3'(1-\pi).$$

Thus, either

$$(15a)\qquad \pi = 1$$

or

$$(15b)\qquad \lambda_3'' = (1-\mu)\lambda_3'; \qquad \lambda_2'' = \mu\lambda_2.$$

If (15b) holds, then we get the result that $\lambda'' = \mu\lambda + (1-\mu)\lambda'$. Furthermore, the nonnegativity of λ_2'' and λ_3'' assure that $0 \leqslant \mu \leqslant 1$.

If $\pi = 1$, then $(\lambda_1/\lambda_1') = (\lambda_2/\lambda_3')$, which implies that $\lambda_1 = \lambda_1'$, the case we set aside earlier and which we now consider. We have $\lambda_1 = \lambda_1'$ and, therefore, $\lambda_2 = \lambda_3'$, which by (13) implies that

$$(16a)\qquad \lambda_1 = \lambda_1' = \lambda_1''$$

and

$$(16b)\qquad \lambda_2 = \lambda_3' = \lambda_2'' + \lambda_3''.$$

However, (16b) says that $\lambda_2'' = \nu\lambda_2$ and $\lambda_3'' = (1-\nu)\lambda_3'$ for some $0 \leqslant \nu \leqslant 1$. Thus $\lambda'' = \nu\lambda + (1-\nu)\lambda'$, and the conditions of case (ii) are satisfied. The conditions of case (ii) must therefore be satisfied if x lies on exactly two contract curves, but is not a bliss point.

Finally, consider the possibility that x lies on all three contract curves. Then there exist $\lambda, \lambda', \lambda''$ such that

$$(17)\qquad \begin{aligned} \lambda_1\nabla_1 + \lambda_2\nabla_2 \qquad\qquad &= 0,\\ \lambda_1'\nabla_1 \qquad\qquad + \lambda_3'\nabla_3 &= 0,\\ \lambda_2''\nabla_2 + \lambda_3''\nabla_3 &= 0. \end{aligned}$$

The system (17) reduces easily to

$$(18)\qquad (\lambda_2\lambda_1'\lambda_3'' + \lambda_1\lambda_2''\lambda_3')\nabla_3 = 0.$$

Consider the possibilities: (a) If $\nabla_3 = 0$, then $x = \gamma$ (Figure 4d, 4e, or degenerate to a point); (b) if $\lambda_2 = \lambda_2'' = 0$, then $x = \alpha = \gamma$ (Figure 4e or degenerate to a point); (c) if $\lambda_2 = \lambda_3' = 0$, then $x = \alpha$ (Figure 4d, 4e, or degenerate to a point); and so forth for the other possibilities. In all cases, the situation in case (iii), or else the degenerate situation in case (iv) is seen easily to hold.

Since x must lie on either zero, one, two, or three contract curves, the proof of the lemma is now completed.

The next lemma states that ϕ is one-to-one except on line segments joining boundary points of S_2. Furthermore, the family of line segments which ϕ collapses to single points must be nonintersecting unless ϕ maps S_2 to a single point (there is no conflict of interest).

LEMMA 3: *Let $\mathscr{G}_i$ be the family of all sets of the form*

$$\{\mu\lambda + (1-\mu)\lambda' : 0 \leqslant \mu \leqslant 1, \lambda_i = \lambda'_i = 0, \lambda \neq \lambda'\}.$$

Let

$$\mathscr{G} = \bigcup_{i=1}^{3} \mathscr{G}_i.$$

Let H be the class of all subfamilies $\mathscr{H}$ of $\mathscr{G}$ having the property that

$$H_1, H_2 \in \mathscr{H} \Rightarrow H_1 \cap H_2 = \phi.$$

Let $R_{\mathscr{H}}$ be the relation defined by identifying a point $p \in S_2$ with the set $H \in \mathscr{H}$ in which it lies; otherwise identify p with the set $\{p\}$.

Then either (i) *there exists $\mathscr{H} \subset H$ such that ϕ is one-to-one on $S_2/R_{\mathscr{H}}$, or* (ii) *$\phi[S_2]$ is a single point.*

PROOF: Lemma 2 shows that if $\phi^{-1}(x)$ is not a single point, then $\phi^{-1}(x)$ is either a line segment H or all of S_2. If $\phi^{-1}(x) = S_2$ for any x, then $\phi[S_2] = x$. If $\phi^{-1}(x_1) = H_1$ and $\phi^{-1}(x_2) = H_2$, and $\lambda \in H_1 \cap H_2$, then $\phi(\lambda) = x_1 = x_2$. Thus, $H_1 \cap H_2$ is empty for $x_1 \neq x_2$. This proves that if (ii) does not hold, then (i) must hold.

THEOREM 1: *$\phi \cdot \pi_R^{-1}$ is a homeomorphism between S_2/R and $\phi[S_2]$, where R is either* (i) *$R_{\mathscr{H}}$ for some $\mathscr{H}$ defined as in Lemma* 3, *or* (ii) *the degenerate relation based on the partition $S_2 = S_2$. If* (i) *holds, and $\mathscr{H}$ is the empty subfamily, then ϕ is a homeomorphism between S_2 and $\phi[S_2]$.*

PROOF: Lemma 3 shows that ϕ is one-to-one on S_2/R for R of type (i) or (ii) above. Furthermore, if $\mathscr{H}$ is the empty subfamily, then $R_{\mathscr{H}}$ is the relation based on the partition $S_2 = \bigcup_{p \in S_2} \{p\}$, in which case $S_2 \cong S_2/R$ and ϕ is one-to-one on S_2. By Lemma 1, ϕ is a continuous function on the compact set S_2. Therefore, $\phi \cdot \pi_R^{-1}$ is a continuous function on the compact set S_2/R. Since a continuous one-to-one mapping is a homeomorphism between its domain and its range, $\phi \cdot \pi_R^{-1}$ is a homeomorphism between S_2/R and $\phi[S_2]$.

Despite the plethora of degenerate and rare cases, it should be stressed that in most situations we are likely to encounter, the contract curves will not cross, $\mathscr{H}$ will be null, and $\phi[S_2]$ will be homeomorphic to the 2-simplex; Figure 3b will prevail.

6.3. *Generalization to 2 Goods, n Individuals*

We have shown that in general the Pareto-optimal set for 2 persons and 3 goods will be homeomorphic to the 2-simplex: the closure of a simple closed curve in 2-space. The generalization to more individuals is rather interesting. The basic result is: for 2 goods and $n > 3$ people, the set of Pareto-optimal points is precisely the union of the Pareto-optimal sets obtained by considering the individuals 3 at a time. Thus, if there are 4 individuals—A, B, C, and D—and if x is Pareto-optimal, then x must be Pareto optimal with respect to (A, B, C), or (A, B, D), or (A, C, D), or (B, C, D). For any Pareto-optimal x, one of the individuals (all but 3) is superfluous.

Graphically, if the contract curves are as shown in Figure 7a, then the Pareto-optimal set is as shown in Figure 7b. Note that every point lies inside at least 1 (in fact, in at least 2) of the 3-person "contract regions." Point Q in Figure 7b, by way of illustration, is Pareto optimal with respect to (A, C, D) taken alone, and also with respect to (B, C, D) taken alone.

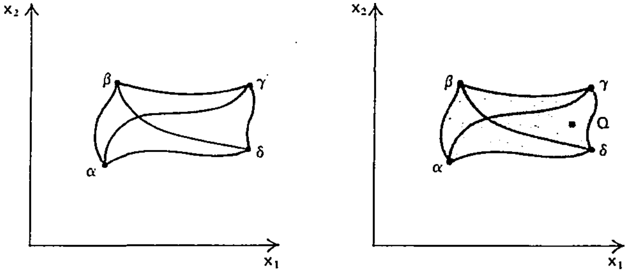

FIGURE 7a

FIGURE 7b

FIGURE 7.—Four-person, two-good case.

The result is a direct consequence of the following theorem in convex analysis, due to Caratheodory (see Rockafeller [6]).

THEOREM 2: *If $\{z_1, \ldots, z_n\}$ are vectors in m-space $(n > m)$ such that*

$$\sum_{i=1}^{n} \lambda_i z_i = 0$$

for some set of nonnegative constants λ_i, then there exists a set of nonnegative constants $\mu_{i_j}(j = 1, \ldots, m + 1)$ and a permutation of indices (i_j) such that

$$\sum_{j=1}^{m+1} \mu_{i_j} z_{i_j} = 0.$$

Letting $z_i = \nabla_i$ (the gradient of the ith utility function at the Pareto-optimal point x), and taking $m = 2$ (the number of goods), we get the result that x must be Pareto optimal with respect to some subset of $m + 1 = 3$ individuals.

We note also that for $n \leqslant 3$, the Pareto-optimal set is, in the typical case, essentially an $(n - 1)$-simplex, i.e., a point if $n = 1$, a curve if $n = 2$, and, as we have shown, a "triangle" if $n = 3$.

6.4. *Generalization to m Goods, n Individuals*

We begin by considering the case $n = m + 1$, where the number of individuals exceeds the number of goods by one. Retain the assumptions about the utility functions $u_i(x)$ that were made earlier in the discussion of the case $m = 2$. It can be shown that typically the Pareto-optimal set is indeed homeomorphic to the m-simplex. Thus, in three dimensions, with four individuals, the Pareto-optimal set is topologically a closed "tetrahedron" whose surface is the union of the three-individual contract regions. The proof is essentially an extension by induction of the result for two dimensions. It is complicated by the proliferation of possibilities of crossing of contract curves, surfaces, and hypersurfaces; these lead to a variety of deviant cases analogous to Figures 4a–4f. Nevertheless, if none of the m-individual "contract hypersurfaces" cross each other, it is easy to see that the mapping is one-to-one, continuous, and, therefore, a homeomorphism from S_m to $\phi[S_m]$, where

$$S_m = \left\{\lambda \in R^m : \sum_{i=1}^{m} \lambda_i = 1, \lambda_i \geqslant 0 \text{ all } i\right\}$$

is the standard m-simplex for $m > 0$.[9]

For $n > m + 1$, Caratheodory's theorem again leads to the result that all Pareto-optimal points must lie on at least one $(m + 1)$-individual "contract hypersurface:" only $(m + 1)$ individuals need be considered in justifying the Pareto optimality of any point. The Pareto-optimal set is, therefore, the union of all of the $(m + 1)$-individual hypersurfaces, each of which is an m-simplex.[10] Another way of viewing this set topologically is as the projection of an $(n - 1)$ simplex into m dimensions. Thus, the region in Figure 6b may be viewed as the projection of a 3-simplex, or tetrahedron, into 2-space ($n = 4$, $m = 2$).

For $n < m + 1$, the Pareto-optimal region is typically an $(n - 1)$-simplex. Thus, for 3 individuals and 3 goods, the Pareto-optimal set is a 2-simplex floating in 3-space, bounded by the locus of tangencies between the indifference surfaces ("spherical shells") of the individuals taken two at a time.

[9] The proof of Theorem A, concerning the dimensionality of the contract set for private goods in the Edgeworth box context, should now be clear. Let w_i^* represent the total endowment of the ith good ($i = 1, \ldots, m$), and let w_{ij} be the amount allocated to the jth individual ($j = 1, \ldots, n - 1$). Let $w_j = (w_{ij})$, the vector of allocations to the jth individual. Then the contract set T is the set of points $(w_1, \ldots, w_{n-1})$ which maximize expressions of the form

$$\sum_{j=1}^{n-1} \lambda_j u_j(w_j) + \left(1 - \sum_{j=1}^{n-1} \lambda_j\right) u_n\left(w^* - \sum_{j=1}^{n-1} w_j\right),$$

where $0 \leqslant \lambda_j \leqslant 1$ ($j = 1, \ldots, n - 1$). By the argument of Lemma 1, the mapping from λ-space to w-space is continuous; it is also clearly one-to-one in the private goods case. Therefore, the result is that T is homeomorphic to S_{n-1}, the $n - 1$ simplex; T is an $(n - 1)$-dimensional figure imbedded in $m(n - 1)$ dimensions.

[10] Unfortunately, no simple generalization of the exact number of "triangles" in which a point may lie seems possible.

TABLE I

GENERAL TOPOLOGICAL PROPERTIES OF n-INDIVIDUAL, m-GOOD PARETO-OPTIMAL SETS

Number of individuals (n)	Number of Goods (m) 1	2	3	4	5	...	$k-1$	k	...
1	S_0	S_0	S_0	S_0	S_0	...	S_0	S_0	...
2	S_1	S_1	S_1	S_1	S_1	...	S_1	S_1	...
3	S_1	S_2	S_2	S_2	S_2	...	S_2	S_2	...
4	S_1	S_2	S_3	S_3	S_3	...	S_3	S_3	...
⋮									
k	S_1	S_2	S_3	S_4	S_5	...	S_{k-1}	S_{k-1}	

Table I summarizes the typical topological nature of the m-good n-individual Pareto-optimal set. Most generally, the k-simplex S_k should be read as S_k/R where R is a relation based on the "collapsing" of a variety of $1, 2, \ldots, (k-1)$-simplices in S_k. Below the solid line are the cases where Caratheodory's theorem applies: these are projections of $(n-1)$-simplices into m-space.

The general result is the following theorem.

THEOREM 3: *The Pareto-optimal set for m public goods and n individuals is typically a q-simplex, S_q. (Otherwise, the set is S_q/R for a suitable equivalence relation R analogous to the possibilities in Theorem 2.) In any case, the dimensionality of the region is $q = \min(m, n-1)$.*

7. SOCIAL CHOICE PROCEDURES AND THE PARETO-OPTIMAL REGION

This theorem provides us with a description of the nature and properties of the public-goods equivalent of the contract curve concept. In a well behaved world with private goods only, we know that free trading on a market will place us on the contract curve. What about social choice procedures—will they place us in the region of Pareto optimality for public goods?

This question does not lend itself readily to analysis. For public goods, there is no pre-eminent outcome-generating procedure equivalent to the market for private goods. This very lack of prominence provides us with some information about our major question. If there were a social choice procedure that were sure to work, we might already be employing it, or at the very least, recognizing its virtues.[11]

One approach would be to examine each of the myriad social choice procedures

[11] See Zeckhauser [10] for some negative conclusions about the efficiency of social choice procedures that allow individuals to give cardinal indications of their preferences (as we get when we ask willingness to pay). Zeckhauser's paper relaxes one of Arrow's conditions, the independence of irrelevant alternatives, but introduces instead the notion that individuals ballot in their self interest and not necessarily honestly.

currently in use to see where and when it might perform efficiently. Here we shall consider four briefly.

7.1. *Party Rule—Optimization for a Restricted Set of Citizens*

This form of government holds a general election, and one party is placed in a position of power. That party is then empowered to select the public-goods bundle during its period of tenure. Say that the party can solve the major problem of this paper and chooses a public-goods bundle that is Pareto optimal for its members. Then so long as the number of members of the party exceeds the number of public goods selected, the analysis of the previous section, as summarized by Theorem 3, demonstrates that the outcome will be Pareto optimal.

The direction of the inference can be reversed. Our result suggests that when observing an efficient outcome in a world of m public goods, it is impossible to determine whether all individuals' preferences were taken into account, or merely those of some particular collection of $m + 1$ individuals.

It is worth noting that the inability to distinguish between the optimal-for-some and optimal-for-all outcomes would not arise if the financing procedure for the public good were unconstrained. For then the optimal outcome for the $m + 1$ party members would leave the remaining members of society with no private goods whatsoever. Party members would incur a negative charge for public goods provision; nonmembers would contribute more than the total cost of provision. Even if there were limits on exploitative possibilities, there would be no difficulty distinguishing between outcomes responsive to the wishes of the many as opposed to just a few. (In the real world, significant constraints on financing possibilities are imposed by our tax systems. Thus sometimes it is difficult to establish or disprove that the general public is the intended beneficiary of party-rule government. Where such constraints can be violated, say, through rigged assessments, it is easy to determine that the rascals are out only for themselves.)

7.2. *Town Meeting—Voting on Issues One by One*

The town meeting stands at the opposite extreme from a party rule situation with efficient internal logrolling. The idealized town meeting calls for a majority rule vote on each issue. There is no trading among issues.

Granting our convexity assumptions, an individual's preferences for any public good will be single peaked. This implies that majority rule along each and every public goods dimension will lead to a determinate outcome at the multi-dimensional median.

Cleary this outcome is Pareto optimal if but one public good is being determined; ruling out ties, $(n - 1)/2$ individuals will want less, an equal number more, and one individual will find things ducky. Turning to the two goods world, the economist's usual analytic playground, if in addition to strict convexity, individuals' preferences have the rather mild property that ordinal ranking of values of one variable does not depend on the level of the other variable, then the town meeting procedure will lead to a Pareto-optimal outcome, as is demonstrated in a footnote.[12] This additional condition is called first-order preferential independence. We say that $u(x_1, x_2)$ displays first-order preferential independence if and only if

$$u(x_1, x_2^*) \geqslant u(x_1', x_2^*) \Rightarrow u(x_1, x_2) \geqslant u(x_1', x_2) \quad \text{for all} \quad x_2$$

and

$$u(x_1^*, x_2) \geqslant u(x_1^*, x_2') \Rightarrow u(x_1, x_2) \geqslant u(x_1, x_2') \quad \text{for all} \quad x_1.$$

It is interesting to note that this result fails to obtain in more than two dimensions. Even much more restrictive conditions will not guarantee that the town meeting approach will lead to a Pareto-optimal outcome. Consider, by way of illustration, three individuals with three goods. Suppose the individuals have bliss points at (1, 2, 4), (2, 4, 1), and (4, 1, 2) respectively, and concentric spherical indifference curves. Then the Pareto-optimal region is the flat triangular region determined by

[12] We shall, for simplicity, assume the number of individuals n to be odd to avoid possible ties. First, note that the strict convexity of preferences guarantees that the collection of preferences along each dimension has the property of single-peakedness. This implies that majority voting on each attribute separately will lead uniquely to the point (m_1, m_2) where m_1 is the *median* of the most-favored levels of the first good, and m_2 is the *median* of the most-favored levels of the second good.

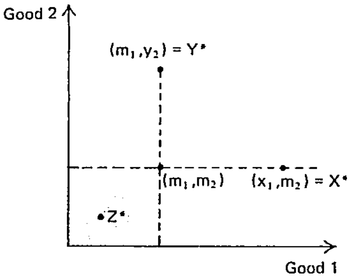

FIGURE A.—Majority voting in two dimensions of public goods.

Figure A displays the situation. It must be the case that m_1 is the first coordinate of the bliss point Y^* of at least one individual; similarly, m_2 must be the second coordinate of the bliss point X^* of at least one individual. Without loss of generality, we assume in Figure A that $m_1 < x_1$ and $m_2 < y_2$; the proof that follows is identical if $m_1 > x_1$ or $m_2 > y_2$ or both.

Now, since m_1 is a median, there must be $(n - 1)/2$ other individuals whose most favored first coordinate is less than or equal to m_1. Similarly, there must be $(n - 1)/2$ other individuals whose most favored second coordinate is less than or equal to m_2. If these two sets of individuals are disjoint, this would account for $(n - 1)$ individuals in addition to the two we already have, for a total of $(n + 1)$. Therefore, at least one individual's bliss point must lie in the shaded region shown in Figure A. Call this point Z^*.

Now we invoke the assumption of preferential independence. The two-person contract curve between X^* and Y^* can cross neither of the dashed lines in Figure B. If it did, say at some point P as shown in Figure B, then a contradiction results. In particular, point R (presumed to be bilaterally Pareto optimal with respect to the two individuals at X^* and Y^*) would be inferior to point Q for both individuals by the independence (and convexity) assumptions. Therefore, the contract curves must appear as in Figure C. It is clear, therefore, that the selected point M lies inside the region bounded by the two-person contract curves of these three individuals. By our Theorem 2, this implies that M is Pareto optimal for the group of n individuals as a whole.

Note that without the assumption of preferential independence, this result would not necessarily hold. In particular, the contract curves could appear as in Figure D.

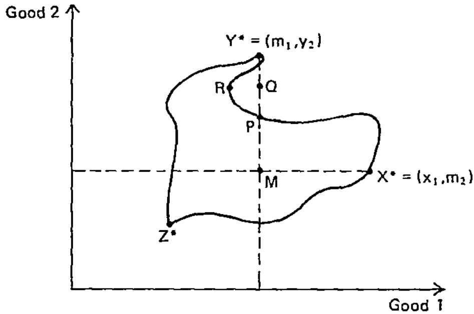

FIGURE B.—Impossible Pareto-optimal region with independence of preferences.

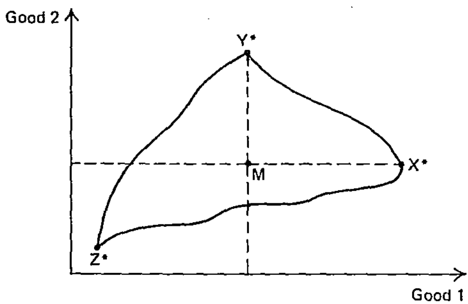

FIGURE C.—Typical Pareto-optimal region with independence of preferences.

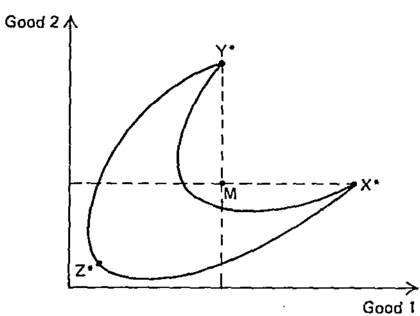

FIGURE D.—Possible situation without preferential independence.

these three points. The median point, (2, 2, 2), does not lie in this region. Observe that this is not an anomaly depending on the fact that there are only three individuals.[13] We could think of three clusters of individuals with bliss points near these three points, and the same result would hold.

7.3. *Legislature—Voting on Issues Combined*

Our traditional legislatures offer neither complete centralized party control, nor simplistic issue-by-issue voting. Delegates are likely party members; sometimes they are faithful, sometimes not. They may vote in blocs that cross party lines. Logrolling, the trading of votes across issues or the combining of different issues into legislative packages, is employed frequently to respond to the different preference intensities that different delegates bring to different issues.

Once two public goods are combined (explicitly, or implicitly through vote trading) for consideration in a single vote, we must expect cyclical majorities to arise. Consider the use of two-issue voting in a situation where the Pareto-optimal region has the typical form shown in Figure 3b. Unlike the town-meeting procedure, voting is now free to move the outcome in diagonal directions. This implies that no point will be a universal victor in paired-majority votes. An outcome in the interior of the shaded area will be defeated by moves to any one of the encircling two-person contract curves. Yet any point along any one of the contract curves can be defeated in a majority vote by a host of points on either of the other two contract curves. (The individual common to both contract curves must improve his position slightly in the move.)

This implies that the single-peakedness condition, even in its multi-dimension form, loses its power to prevent cyclical majorities once two or more issues can be combined in a single vote.[14] (Only ridiculously strong conditions, such as that the bliss point of one voter lies on the contract curve of the others, will prevent cyclicality.) The reassurance that some analysts of social choice have created for themselves by projecting the relative ubiquity of single-peaked phenomena seems somewhat overextended. This observation is reinforced once we recognize that individual public goods are in general characterized by many attributes. Thus even single-issue majority must be expected to lead to intransitive social-preference orderings.

The relatively infrequent appearance of cyclical majorities in functioning legislatures cannot be explained by some geometric property of individuals' preferences. As this paper has stressed, matters such as procedural rules and institutional constraints provide key insights into real-world social-choice phenomena.

[13] Here, as we have found elsewhere, say with the Arrow impossibility theorem or the integrability conditions of demand theory, matters may go smoothly with two goods or alternatives, but they may break down when we move up to three.

[14] See Arrow [2, pp. 75–80] and Sen [9, pp. 166–168] for a discussion of the power of the single-peakedness criterion when voting takes place along a single dimension.

7.4. A Generalized Lindahl Procedure

The model in the paper is closely related to the Lindahl procedure for selecting a single public good (see Lindahl [4]). The finance mechanisms assumed in both models are essentially the same—individuals are assumed to pay for the public good according to a constant proportional tax rate. The key difference between them is that while our model assumes the tax shares to be prespecified and immutable, the emphasis of the Lindahl procedure is on adjusting the tax shares in such a way as to achieve unanimity on the preferred level of provision of the good.

The Lindahl procedure, and the analogy with our model, extends to the case of more than one public good. The extended Lindahl procedure seeks $n \times m$ tax shares δ_j^i (for the ith individual and the jth good, subject to $\Sigma_{i=1}^{n} \delta_j^i = 1$ for all j) such that the "mechanism-constrained" bliss points of all individuals coincide. It can be shown that, in the m-good case, the extended Lindahl procedure yields a unique solution. Let $u^i(x, w^i)$ be the utility function for the ith individual, where $w^i = B_0^i - \Sigma_{j=1}^{m} \delta_j^i x_j$ is the residual wealth available for the purchase of private goods. Then the Lindahl conditions are

$$u_j^i - \delta_j^i u_w^i = 0 \qquad (i = 1, \ldots, n; j = 1, \ldots, m),$$

and

$$\sum_{i=1}^{n} \delta_j^i = 1 \qquad (j = 1, \ldots, m).$$

These $m(n + 1)$ equations determine (under the assumed convexity conditions) the $m(n + 1)$ variables $\{\delta_j^i\}$ and $\{x_j\}$. In terms of our model, then, the "mechanism-constrained" Pareto-optimal region shrinks to a single point if the Lindahl tax shares are in effect. Furthermore, that point is Pareto optimal in the global sense (i.e., if the tax shares are allowed to vary) as well as in the "mechanism-constrained" sense.

Other analysts, indeed proponents of various social-choice procedures, will wish to trot out their own mechanisms to see whether they can generate efficient public-goods bundles. The results with our three procedures are at the same time both encouraging and sobering. It was good to see that the Pareto region was attainable, but discouraging to note that neither of the successful procedures much resembles the elaborate mechanisms the real world employs to select public goods.[15]

8. CONCLUSION

Our analysis reveals to us four points. First, it is possible and profitable to consider the structure of the Pareto-optimal region in a world with public goods. Second, the intuitively appealing conjecture that this region will be enclosed by the contract loci for different collections of individuals is borne out. Third, some

[15] Thomas Schelling notes a frequently overlooked point. Pareto optimality may be a meaningful concept when it is achieved. However, non-optimality tells us little, for we get no indication of how far, in terms of either value or distance, we are from the region of desirability.

recognized social choice procedures can be expected to lead us to the Pareto-optimal region; others can not. Still others will lead us to make cyclical choices. Fourth, a study of institutional procedures would seem to be a profitable way to gather further insights on patterns of public goods provision.

Harvard University

Manuscript received March, 1972; last revision received July, 1973.

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