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CRITICAL RATIOS AND EFFICIENT ALLOCATION

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1. Introduction

Many tributaries have contributed to the development of public sector economics. But the mainstream brims full of unifying elements and analytic structures of general applicability. (This commonality, indeed, was one inspiration for the founding of this journal.) This paper demonstrates an equivalence among three ostensibly different problems, namely: (1) the efficient division of bundles of goods to which individuals attach value, (2) the selection of public projects under a constrained budget, and (3) the Neyman–Pearson problem of hypothesis testing in statistics. The first may be recognized as a variant of the classical problem of comparative advantage. As constrained optimization problems, the latter two turn out to have exactly the same underlying mathematical structure. For the three together, an analogy extends to a paradigmatic unconstrained optimization problem.

Each of the three is discussed below in its simplest (binary) form, with a numerical example. The well-known nature of the optimal solution is illustrated; the unifying feature is that all three solutions involve the computation of a 'critical ratio' which acts as a cutoff point for allocation. The underlying constrained optimization formulation is then given, and in each case it is seen to be identical to a standard reference problem to be described presently. In the next section, the associated unconstrained problem for each of the examples is given, and the multipliers are interpreted as the numerator and denominator of the 'critical ratio'. In the last section, the results are generalized to the case of allocation to n categories, rather than two.

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2. The generic constrained optimization problem: binary case

The mathematical structure of the underlying optimization problem is as follows. Let x_i^k be given for $i = 1, ..., M$, and $k = 1, 2$. We wish to choose $\delta_i (i=1, ..., M)$ to maximize

$$J = \sum_{i=1}^{M} \delta_i x_i^1 \tag{1}$$

subject to the constraints

$$\sum_{i=1}^{M} \delta_i x_i^2 \leqslant R \tag{2}$$

and

$$0 \leqslant \delta_i \leqslant 1 \ (i=1, ..., M) \, . \tag{3}$$

The optimal δ_i^* are always given by a solution of the form

$$\delta_i^* = \begin{cases} 1 \text{ if } \dfrac{x_i^1}{x_i^2} > \lambda \\ 0 \text{ if } \dfrac{x_i^1}{x_i^2} < \lambda \\ \pi \text{ if } \dfrac{x_i^1}{x_i^2} = \lambda \end{cases} \tag{4}$$

where λ and π are chosen so that the constraint (2) is met with equality. The number λ is called the *critical ratio* for the constrained optimization problem.

3. The efficient division problem

Suppose we have a bundle of M goods and suppose there are two individuals, A and B, who attach numerical values to each of the goods.

The total value accruing to an individual is the sum of the values (to him) of the goods he receives. We may therefore think of the preference function of an individual as additive with respect to the numeraire in which his value of the goods is measured. The objective is to divide the goods between A and B to maximize the satisfaction of A, subject to the condition that B receives some particular fraction of his total valuation of the entire bundle. The variant of this problem where it is stipulated that each player must receive at least half of his total valuation is traditionally referred to as the fair division problem. [1]

Equivalent schemes of allocation are found in the public sector when facilities or resources are assigned to particular purposes or projects. For example, we may wish to divide recreational forest lands between hunting and camping to maximize the enjoyment of hunters, subject to campers receiving some specified minimum level of pleasure. Each allocable tract offers particular levels of enjoyment for each group.

In order to keep this general problem mathematically tractable, it is useful to assume that, in one way or another, the goods are completely divisible. One alternative is to suppose that, in fact, the goods are completely divisible. Another alternative is to assume that the goods are not divisible, but that there are sufficiently many goods that each one is small in value compared to the whole; thus, the bundle is approximately a continuum. A third alternative, which will be adopted here since it applies most naturally to the hypothesis testing problem to be discussed presently, is to assume that the values attached to the goods by the individuals are in fact cardinal utilities deriving from an additive Von Neumann–Morgenstern preference function. In this context, we further assume that lotteries on goods are permitted.

As an example, suppose table 1 gives the utility values of A and B for five commodities. (Ignore for now the headings enclosed in boxes; these will be used in a later example.) Suppose that we require that B must

[1] See Steinhaus (1948). Unless the participants place the same relative valuations on all goods, there will always be somewhat of a surplus to be distributed. A variety of schemes is available to provide a fair division of this surplus. One that requires no common external numeraire unit for the players' valuations is to maximize the minimum fraction of total valuation received by any player. In an n player game, all will receive the same fraction which will be greater than or equal to $1/n$.

An analogous problem arises when M tasks are to be divided between two individuals who can perform each of the tasks in varying amounts of time. The fair division aspect enters if we stipulate that neither individual shall be required to contribute more than half of the total number of man-hours spent on the tasks.

Table 1
Efficient division problem (project selection problem).

		Commodity (project)				
Individual	(Project characteristic)	1	2	3	4	5
A	(Net benefit)	20	60	100	40	30
B	(Resource used)	20	30	40	10	40
Ratio	(Ratio)	1.0	2.0	2.5	4.0	0.75

receive at least half of his total valuation of the goods.[2] The optimal allocation is determined as follows. First, calculate the *ratio* of A's evaluation to B's evaluation of each of the goods, and order the goods from the highest to the lowest ratio. Thus, as in fig. 1, the goods are ordered along a spectrum of ratios. Next, we determine a *critical ratio* λ such that when all goods with ratio greater than λ are allocated to A, all goods with ratio less than λ are allocated to B, and all goods with ratio equal to λ are allocated by lottery with a probability π that A gets the good, then B gets just half of his evaluation of the total bundle. In the example of fig. 1, half of B's total is 70, and is obtained exactly with a critical ratio of $\lambda = 2.0$, and $\pi = 2/3$. This allocates goods 1 and 5 to B, allocates goods 3 and 4 to A, and provides for a lottery for good 2 with A getting a probability of 2/3 of winning. A's satisfaction under this scheme is 180, and this is the constrained maximum.

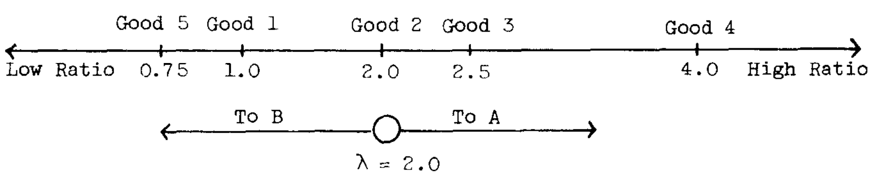

Fig. 1. Ordering of goods by ratio.

[2] In the generic form of the previous section, the x_i^k are the valuations of the ith good by the kth individual; $\delta_i = 1$ if A gets the ith good, $\delta_i = 0$ if B gets the ith good, and $\delta_i = \pi$ if there is a lottery for the ith good with A getting probability π. The value of R in (2) is $\Sigma_{i=1}^{M} x_i^2$.

4. The project selection problem

A governmental agency has a fixed budget (or other limited resource), and must choose from among M independent projects. The agency may select as many of the projects as it wishes, provided that the resource constraint is not violated. The objective is to maximize total net benefits. (To avoid the use of lotteries, which is clearly inappropriate in this context, and to avoid the problem of indivisibilities, we shall assume that all projects are completely divisible with constant returns to scale.)

Table 1, in the context of the headings enclosed in parenthesis, presents an example with five available projects, and a resource constraint of 70. The optimal selection is determined by calculating the ratio of net benefit to resource used for each project, and determining a critical ratio λ such that when all projects with ratio greater than λ are undertaken, all projects with ratio less than λ are not, and all projects with ratio equal to λ are undertaken in some fraction π, the resource is just exhausted. In the above example, the critical ratio $\lambda = 2.0$, so that projects 3 and 4 are undertaken, and the residual is made up by implementing $\pi = 2/3$ of project 2.

In the generic form of the problem, x_i^1 is the net benefit from the ith project, and x_i^2 is the amount of budget consumed by the ith project. The value of δ_i represents the fraction of the ith project undertaken, and the constraint value R is the limit on the constrained resource (or budget). The nature of the solution, involving a critical ratio is, of course, identical. It is evident that this variant of the generic problem also relates to cost/effectiveness procedures. The task may be to select projects in such a way to minimize expenditures subject to meeting a certain level of total performance. Here the constraint is the performance level, which need not be measurable in the dollar units. Conversely, the objective in this form of problem could be to maximize effectiveness measured in some non-commensurable units subject to some ceiling on dollar expenditure.

5. The Neyman–Pearson hypothesis testing problem

The most striking historical example of independent derivation of results in a completely different context is the classical Neyman–Pearson theory of hypothesis testing. The classical formulation of the problem is as follows. Let there be two 'states of the world', θ_1 and θ_2. We are to perform an experiment which has M possible outcomes – $E_1, ..., E_M$ – each of which has a pair of conditional probabilities $P\{E_i|\theta_k\}$ $(i=1, ..., M; k=1, 2)$ corresponding to the two states of the world. After observing the result of the experiment, we must 'guess' at the true state of the world. Denote a guess of θ_1 by G_1 and a guess of θ_2 by G_2. We are said to make a 'type 1 error' if we guess G_1 when θ_2 is true, and we are said to make a 'type 2 error' if we guess G_2 when θ_1 is true. Our objective, in the classical formulation, it to minimize the (conditional) probability of a type 2 error, $P\{G_2|\theta_1\}$, subject to the constraint that the probability of a type 1 error, $P\{G_1|\theta_2\}$ is at most some number α. Alternatively, we wish to maximize $P\{G_1|\theta_2\}$, subject to the constraint $P\{G_2|\theta_2\} \geqslant 1 - \alpha$.

The well-known solution is to calculate the *likelihood ratios* for each possible outcome E_i,

$$L_i = \frac{P\{E_i|\theta_1\}}{P\{E_i|\theta_2\}}$$

and to compute a *critical ratio* λ such that if we choose G_1 when $L_i > \lambda$, G_2 when $L_i < \lambda$, and randomize between G_1 and G_2 with some probability π when $L_i = \lambda$, the value of $P\{G_2|\theta_2\}$ is just equal to its minimum permitted value of $1 - \alpha$. That is, we choose

$$P\{G_1|E_i\} = \begin{cases} 1 \text{ if } L_i > \lambda \\ 0 \text{ if } L_i < \lambda \\ \pi \text{ if } L_i = \lambda \end{cases}$$

with λ and π chosen such that $P\{G_2|\theta_2\} = 1 - \alpha$.

To see that the character of the classical solution is no surprise and that, in fact, this problem is structurally identical to the others, consider the following formulation. Our objective is to maximize

$$P\{G_1|\theta_1\}$$

subject to

$$P\{G_2|\theta_2\} \geqslant 1 - \alpha .$$

Expanding the probabilities conditionally on the E_i, our objective is to maximize

$$\sum_{i=1}^{M} P\{G_1|E_i\}P\{E_i|\theta_1\} \tag{5}$$

subject to

$$\sum_{i=1}^{M} P\{G_2|E_i\}P\{E_i|\theta_2\} \geqslant 1-\alpha \,. \tag{6}$$

If we let $x_i^k = P\{E_i|\theta_k\}$ and $\delta_i = P\{G_1|E_i\}$ and $R = \alpha$, then (5) and (6) are identical to (1) and (2). The solution is therefore identical to (4), as proved by Neyman and Pearson several decades ago.

6. Interpretations of the critical ratio, λ

Consider the generic problem illustrated by (1)–(3). The same solution (4) would be obtained if we replace (1) and (2) by an unconstrained maximand

$$J' = \sum_{i=1}^{M} \delta_i x_i^1 + \lambda \sum_{i=1}^{M} (1-\delta_i)x_i^2 \,, \tag{1'}$$

where λ is identical to the critical ratio given in (4).[3] The value of λ is thus seen to have three interpretations: (i) as the critical ratio in the solution to the generic constrained problem, (ii) as the appropriate ratio of weights in an unconstrained maximization problem with the same solution as the constrained problem, and (iii) as the shadow price of the constraint (2) with respect to the objective function (1) in the constrained version.

In the division-of-goods problem, by varying λ from zero to infinity (or, equivalently, by varying the lower bound on the amount of satisfaction individual B must receive), we generate the full set of Pareto-

[3] In the case $\lambda = \infty$, the appropriate maximand is $J' = \Sigma_{i=1}^{M} (1-\delta_i)x_i^2$.

optimal allocations between the two individuals. A high value of λ reflects a relatively high valuation of individual B, or, equivalently, a high shadow price (in terms of A's utility) for the constraint on B's satisfaction. Where the allocation problem is of resources or facilities to projects or groups, λ represents the valuation ratio placed on the outputs of the two projects, or welfares of the two groups.

In the project-selection or cost-effectiveness problems, λ represents the shadow price of the constrained resource in terms of net benefits foregone. If the constrained resource is valued at λ times the value of net benefits, then the same solution would be obtained if the constraint were removed and the expression (1′) were maximized. The resulting allocation, for any value of λ, is efficient if a unit of resource has a cost of λ units of net benefits.

In the hypothesis testing problem, for any level α, the corresponding optimal decision procedure is known as the 'best level α test'; that is, for any limit on type 1 error, the resulting test minimizes type 2 error. Such a test is also 'efficient' in the following sense. Suppose we begin with Bayesian prior probabilities π_1 and π_2 that the states θ_1 and θ_2, respectively, obtain. If our objective is to minimize the probability of error (aggregated by our prior beliefs) then we have an unconstrained objective function

$$J' = \sum_{i=1}^{M} P\{G_1|E_i\}\, P\{E_i|\theta_1\}\, P\{\theta_1\} + \sum_{i=1}^{M} P\{G_2|E_i\} P\{E_i|\theta_2\}\, P\{\theta_2\}$$

$$= \pi_1 \sum_{i=1}^{M} \delta_i x_i + \pi_2 \sum_{i=1}^{M} (1-\delta_i) x_i^2 \,,$$

which is analogous to (1′) with λ replaced by the ratio of the prior probabilities, π_2/π_1. The notion of 'efficiency' in the statistical context is replaced by the statistical notion of 'admissibility', and the unconstrained weighted objective function is precisely the 'Bayes risk' of classical statistics. If our losses[4] associated with type 1 and type 2 errors were not equal, then the appropriate weight λ would be the product of the ratio of the priors and the ratio of the losses.

[4] If $V(G_i|\theta_j)$ is our payoff if we guess i and if j is true, then the 'losses' are $V(G_2|\theta_1) - V(G_1|\theta_1)$ and $V(G_1|\theta_2) - V(G_2|\theta_2)$ respectively.

7. Generalization to higher dimensions

Consider the following generalization of our basic problem. Let x_i^k be given for $i = 1, ..., M$, and $k = 1, ..., N$. We wish to choose δ_i^k (i=1, ..., M; k=1, ..., N) to maximize

$$J = \sum_{i=1}^{M} \delta_i^1 x_i^1 \tag{7}$$

subject to the constraints

$$\sum_{i=1}^{M} \delta_i^k x_i^k \geqslant R_k \quad (k=2, ..., N)\ , \tag{8}$$

$$\sum_{k=1}^{N} \delta_i^k = 1 \qquad (i=1, ..., M)\ , \tag{9}$$

and

$$0 \leqslant \delta_i^k \leqslant 1 \qquad (i=1, ..., M; k=1, ..., N)\ . \tag{10}$$

(In the binary case, $\delta_i^1 = \delta_i$, $\delta_i^2 = 1 - \delta_i$, and $R_2 = \Sigma_{i=1}^M x_i^2 - R$, so that (7)–(10) reduce to (1)–(3).)

The optimal δ_{i*}^k are always given by a solution of the following form. Define N numbers θ_i, and form critical ratios $\lambda_{ij} = \theta_i / \theta_j$.[5] The rule is to set

$$\delta_{i*}^k = \begin{cases} 1 \text{ if} \dfrac{x_i^k}{x_i^j} > \lambda_{kj} \text{ for all } j \neq k \\ 0 \text{ if} \dfrac{x_i^k}{x_i^j} < \lambda_{kj} \text{ for any } j \neq k \\ \pi_i^k \text{ if} \dfrac{x_i^k}{x_i^j} \geqslant \lambda_{kj} \text{ for all } j \neq k \\ \quad \text{with equality for some } j \neq k\ , \end{cases} \tag{11}$$

[5] A more general, though less transparent, formulation by which the N^2 parameters λ_{ij} are selected directly subject to the conditions $\lambda_{ij}\lambda_{jk} = \lambda_{ik}$ would eliminate the difficulties which occur when $\theta_i = \theta_j = 0$ and $\theta_i = \theta_j = \infty$.

where the λ_{kj} and π_i^k are chosen so that the constraints (8) are met with equality.

This procedure is best clarified by an example. Consider the problem of dividing six goods among three individuals – A, B, and C. Let table 2 give the Von Neumann–Morgenstern (independent) utilities attached to the goods by the individuals. Thus, for example, A's utility for good #3 is 50. Suppose the objective is to maximize A's total utility, subject to the constraints that B and C each get at least one-third of their respective valuations of the entire bundle. In this case, this implies that B and C must each receive at least 50 units of utility. The critical ratios which 'clear' the problem are $\lambda_{AB} = 1.5$, $\lambda_{AC} = 1.5$, and $\lambda_{BC} = 1.0$. The optimal allocation is as follows:

A gets 3, 4, 5, and a $\frac{1}{3}$ chance at 2 ,

B gets a $\frac{1}{2}$ chance at 1 and a $\frac{2}{3}$ chance at 2 ,

and

C gets 5 and a $\frac{1}{2}$ chance at 1 .

Thus, A gets a (maximum possible) total of 220, B and C each get their minimum allowable totals of 50.

Intuitively, what is going on here is exactly the same mechanism as we find in international exchange. The analogy between the λ_{ij} in the above formulation and exchange ratios (or rates) between pairs of countries is exact. As in international exchange, the model is equilibrated by raising or lowering various exchange rates until the market clears.

Table 2
Three-person allocation problem.

Individual	1	2	3	4	5	6
A	20	90	50	80	30	60
B	20	60	10	40	15	5
C	20	40	20	20	40	10
Ratio $_{AB}$	1.0	1.5	5.0	2.0	2.0	12.0
Ratio $_{BC}$	1.0	1.5	0.5	2.0	0.375	0.5
Ratio $_{AC}$	1.0	2.25	2.5	4.0	0.75	6.0

The interpretations of the multidimensional case in each of the four areas of application are straightforward. In the allocation of goods problem, the allocation is now among N, rather than two, individuals. In the hypothesis testing problem, there are N, rather than two, competing hypotheses. In the project selection problem, the formal analogy is not quite as exact. The best analogy is that there are $N-1$ constrained resources to which we attach shadow costs such that when we do projects with positive *net* value and reject projects with negative *net* value, all resources with positive shadow costs are just exhausted.

As in the binary case, the constrained version of the problem has an associated unconstrained version. In the general case, the unconstrained objective function is the weighted sum of N components; the weights are in the same ratios as the shadow prices of the constraints (λ_{1k}). The interpretations of the weights in each of the four problems are direct extensions of the interpretations in the binary case.

8. Summary

It has been shown that the problems of efficient division of goods, selection of public projects with a constrained resource, and classical hypothesis testing all have the same underlying mathematical structure. Their solutions are exactly the same: they all involve the use of critical ratios as cutoff points for allocation, and for essentially the same underlying reasons. In all cases the interpretation of the critical ratios as shadow prices of the constraints, and as multipliers in an associated unconstrained problem, is similar. The particular interpretation of the multipliers in the hypothesis testing problem as Bayes prior probabilities is of special interest.

Reference

Steinhaus, Hugo, 1948, The problem of fair division, Econometrica 16, 101–104.