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ANOTHER TYPE OF RISK AVERSION

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This paper introduces the concept of size-of-risk aversion, discusses its mathematical properties, and relates it to other measures of risk aversion. Decision makers whose utility functions display size-of-risk aversion will pay greater percentage premiums over fair actuarial value to insure against risks of greater magnitude.

1. INTRODUCTION

LET $u(w)$ be a utility function for wealth, w. Pratt [**2**] has shown that $r(w) = -u''(w)/u'(w)$ can be interpreted as a measure of local risk aversion at a particular wealth position. He shows that if r is decreasing with w, the risk premium (expected monetary value minus cash equivalent) for a particular lottery will be a decreasing function of prelottery wealth. Pratt makes the normative observation that many decision makers would feel they ought to pay less for insurance against a given risk, the greater their assets. Such a decision maker would want to choose a utility function for which $r(w)$ is decreasing.

In contemplating their willingness to pay insurance premiums, some decision makers might find it easier to define their preferences in a somewhat different context. Consider an individual faced with what we shall call a simple lottery, a lottery with but two payoffs, one of which is zero. This lottery will result in a loss λ (gain γ) with probability p, but no change in wealth with probability $1 - p$. Given his current wealth, how would the insurance premium (certainty equivalent) he would pay (accept) to avoid the lottery vary with the size of λ?

2. SIZE-OF-RISK AVERSION

We would expect that many individuals would be willing to pay a larger premium as a percentage of the fair actuarial value of an unfavorable simple lottery, the greater is the magnitude of the loss included in the lottery. For example, if such an individual would be willing to pay \$130 to insure against a .1 probability of a loss of \$1,000, he would pay more than \$260 to insure against a \$2,000 loss with the same probability. In some sense, then, these individuals are more averse to risk, the greater is the size of the potential loss.

Similarly, they may also feel that the certainty equivalent of favorable simple lotteries is a proportionally decreasing function of the gain γ. Such behavior will be called size-of-risk aversion.

More precisely, let W be the initial wealth position. Let w represent total possible assets, so that only positive values need be considered. Let

$$(1) \qquad u(W - y) = p \cdot u(W - \lambda) + (1 - p) \cdot u(W)$$

define y, the insurance premium one is just willing to pay to insure against a loss of

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size λ that is incurred with probability, p, $0 < p < 1$. A utility function is size-of-risk averse for losses if y/λ is an increasing function of λ, for all $\lambda > 0$.

Similarly, let

$$p \cdot u(W + \gamma) + (1 - p)u(W) = u(W + y^*) \tag{2}$$

define y^*, the certainty equivalent one is just willing to accept in place of a chance p to gain γ. A utility function is size-of-risk averse for gains if y^*/γ is a decreasing function of γ.

THEOREM 1: *Let* $s(x; W) = x[-u''(W + x)/u'(W + x)]$. (i) *If* $s(x; W)$ *is increasing for* $m \leqslant x \leqslant 0$, *then* $u(w)$ *is size-of-risk averse for losses less than* $-m$. (ii) *If* $s(x; W)$ *is increasing for* $0 \leqslant x \leqslant M$, *then* $u(w)$ *is size-of-risk averse for gains less than* M.

PROOF:[2] Set

$$f(x) = u(W + x) - u(W). \tag{3}$$

Then $f'(x) = u'(W + x); f''(x) = u''(W + x)$.

After substitution, (1) becomes

$$f(-y) = p \cdot f(-\lambda). \tag{4}$$

By differentiating y/λ with respect to λ, we find

(5) y/λ is an increasing [nondecreasing, decreasing] function of $\lambda > 0$, if and only if $y' > [\leqslant, <] y/\lambda$, for all λ.

We differentiate the log of both sides of (4) to get

$$\frac{f'(-y)y'}{f(-y)} = \frac{f'(-\lambda)}{f(-\lambda)}. \tag{6}$$

From (5) and (6), we must show

$$\frac{f'(-y)y}{f(-y)} > \frac{f'(-\lambda)\lambda}{f(-\lambda)}. \tag{7}$$

Consider λ fixed. Then y is a function of p. Inequality (7) holds whenever

$$N(p) = f'(-y)yf(-\lambda) - f'(-\lambda)\lambda f(-y) > 0. \tag{8}$$

Clearly, $N(0) = N(1) = 0$. If $N(p)$ is concave, then it must be positive for $0 < p < 1$ and part (i) of the Theorem will be proved.

$$N'(p) = f(-\lambda)[-f''(-y)y'y + f'(-y)y'] + f'(-y)y'f'(-\lambda)\lambda. \tag{9}$$

But differentiating (1) with respect to p, we get $-u'(W - y)y' = u(W - \lambda) - u(W)$, so

$$y' = -\frac{f(-\lambda)}{f'(-y)}. \tag{10}$$

[2] This proof employs the method of Mossin [**1**].

Thus from (9) and (10),

$$N'(p) = [f(-\lambda)]^2\left[\frac{f''(-y)y}{f'(-y)} - 1\right] - f(-\lambda)f'(-\lambda)\lambda.$$

By assumption, $f''(-y)y/f'(-y) = s(-y, W)$ is increasing with $-y$. As p increases, $-y$ decreases, so $N'(p)$ is decreasing. Thus $N(p)$ is concave. Part (ii) can be proved similarly.

3. EXAMPLES OF UTILITY FUNCTIONS WITH SIZE-OF-RISK AVERSION

Many simple, familiar, utility functions such as $u(w) = \log(w)$, $u(w) = -e^{-w}$, and $u(w) = w^q$, $0 < q < 1$, all have $u'(w) > 0$, $u''(w) < 0$, and $s(x; W)$ increasing for all positive values.

Changing the scale on which money is measured should not affect decisions. The following theorem proves that linear operations on the argument of a size-of-risk averse utility function preserve size-of-risk aversion.[3]

THEOREM 2: *Let $a > 0$; $u_1(x) = u(ax + b)$ is size-of-risk averse for $x_0 \leqslant W \leqslant x_1$ if and only if $u(x)$ is size-of-risk averse for $ax_0 + b \leqslant W \leqslant ax_1 + b$.*

PROOF:

$$s_1(x; W) = x\left[\frac{-u_1''(W + x)}{u_1'(W + x)}\right] = x\left[\frac{-u''(a(W + x) + b)\cdot a}{u'(a(W + x) + b)}\right]$$

$$= ax\left[\frac{-u''((W + b) + ax)}{u'((W + b) + ax)}\right] = s(ax; W + b).$$

This theorem is the equivalent of Pratt's Theorem 3, which he uses to help him find examples of functions that display risk aversion.

In general, composites and sums of size-of-risk averse functions are not size-of-risk averse. Consider the frequently employed utility function $u(w) = -a^{-bw} - c^{-dw}$ with $a, b, c, d > 0$ and $b > d$. It can be demonstrated that $s(x; W)$ is increasing for $0 \leqslant W + x \leqslant W + (b^2/(b - d)^2)$. In addition, it is increasing for all positive $W + x$ if

$$\frac{b - d}{b}W + \frac{b}{b - d} \geqslant \log\frac{ab(b - d)}{cd^2}. \tag{11}$$

4. RISK AVERSION WITH RESPECT TO SCALE OF LOTTERY

At times an individual may be faced with a lottery that gives him a positive probability of a number of different outcomes. It may be of interest to know how the amount that he will pay to avoid or will accept to give up the lottery will vary with a scale parameter that multiplies all the possible payoffs.

[3] It is even simpler to show that $s(x; W)$, like Pratt's measures of risk aversion, is not affected by linear transformations of the utility function.

If the risk premium $\pi(W, c\tilde{z})$ as a proportion of c is to increase with c, for an arbitrary fair gamble $\tilde{z}$, it suffices to have $s(x; W)$ increasing for all W in $[0, +\infty)$. This condition is somewhat stronger than the hypotheses of the theorem below and is satisfied, for example, by $u(w) = -e^{-kw}$.

Let $E(\tilde{z}) = 0$ and suppose π is defined by

$$u(W - \pi(W, c\tilde{z})) = E(u(W + c\tilde{z})). \tag{12}$$

THEOREM 3: *$\pi(W, c\tilde{z})$ is an increasing [decreasing] function of c if*

$$w\frac{u''(w)}{u'(w)} \quad \textit{and} \quad \frac{-u''(w)}{u'(w)}$$

are nonincreasing [nondecreasing] functions of w.

PROOF: Normalize $\tilde{z}$ by multiplying by W/c. Then by Theorem 6 of Pratt, since $z(-u''(z)/u'(z))$ is nondecreasing

$$\frac{\pi(W, W\tilde{z})}{W} \leqslant \frac{\pi(W + \varepsilon, (W + \varepsilon)\tilde{z})}{W + \varepsilon}. \tag{13}$$

By Theorem 2 of Pratt, since $(-u''(z)/u'(z))$ is nonincreasing

$$\frac{\pi(W + \varepsilon, (W + \varepsilon)\tilde{z})}{W + \varepsilon} \leqslant \frac{\pi(W, (W + \varepsilon)\tilde{z})}{W + \varepsilon}. \tag{14}$$

If $-u''(w)/u'(w)$ is nonincreasing, then for $w > 0$, $w(u''(w)/u'(w))$ is decreasing. If $w(u''(w)/u'(w))$ is nonincreasing then for $w < 0$, $-u''(w)/u'(w)$ is decreasing. Thus the inequality in either (14) or (15) must be strict. We combine (14) and (15) to complete the proof.

This result can be extended to nonneutral gambles. Suppose $E(\tilde{z}) = \mu$. Then $E(u(W + c\tilde{z})) = E(u(W + c\mu + c(\tilde{z} - \mu)))$, so

$$u(W - \pi(W, c\tilde{z})) = u(W + c\mu - \pi_1(W + c\mu, c(\tilde{z} - \mu))). \tag{15}$$

Since $E(\tilde{z} - \mu) = 0$, π_1/c is increasing, so $\pi(W, c\tilde{z}) = (-c\mu + \pi_1)/c$ is increasing.

5. CHOICE OF THE UTILITY FUNCTION

There is a direct relationship between size-of-risk aversion, risk aversion, and $r^*(w) = wr(w)$, which Pratt shows to be a measure of proportional risk aversion. The relationship can be summarized by

$$s(x; W) = r^*(W + x) - Wr(W + x) \tag{16}$$

and

$$\frac{ds}{dw} = x\frac{dr}{dw} - r. \tag{17}$$

Thus an individual has size-of-risk aversion if the difference between his proportional and his absolute risk aversion (measured in units of initial wealth) is increasing.[4] Constant risk aversion utility functions display size-of-risk aversion, as do decreasingly risk-averse utility functions that have nondecreasing proportional risk aversion.[5]

Individuals may wish to choose utility functions to represent their preferences that display risk aversion or size-of-risk aversion. If the former concept is more general and tractable, the latter may be better understood on an intuitive basis. From a normative standpoint, we believe that consideration of both kinds of risk aversion should be part of the process through which a utility function is chosen.

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REFERENCES

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[2] PRATT, JOHN W.: "Risk Aversion in the Small and in the Large," *Econometrica*, 32 (January–April, 1964), pp. 122–136.

[4] It can be shown that for very small losses or gains, any risk-averse utility function that has u''' continuous or locally bounded will display size-of-risk aversion.

[5] The utility function $u(w) = -e^{1/w}$, which is decreasingly risk-averse, shows that this condition alone is not sufficient. Measuring w in units of \$100,000, if initial wealth is \$10,000, there will not be size-of-risk aversion for simple lotteries offering gains of more than \$15,000.